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VECTOR THEORY OF ULTRASONIC IMAGING

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Abstract

Sofar works on ultrasonic diffraction imaging are based on scalar theory of sound wave. This is not correct as sound has vector nature. When sound propagates in fluids it can be approximated as a scalar wave as there is no polarization. But when sound propagates in solids, its vector nature has to be considered as polarization occurs and transverse wave as well as longitudinal wave will appear. Vector theory is especially needed when the obstacle size is smaller than the wavelength. We use the Smythe-Kirchhoff approach for the vector theory of diffraction.Comparing the result with the scalar Kirchhoff approximation, we find that both contain the same diffraction distribution factor and the same dependence on wave number. But the scalar result has no azimuthal dependence whereas the vector expression does. The azimuthal dependence variation comes from the polarization properties of the field and must be absent in a scalar approximation. We use the analogy of the sound velocity as equivalent to the magnetic field and the acoustic stress field as equivalent to the electric field to convert our result from the electromagnetic case to the acoustic case. We then derive the image formation theory based on the vector diffraction theory. We use the angular spectrum approach. We found the existence of the components of the angular spectrum known as evanescent waves. These waves are more properly treated in a vectorial approach. We then discuss the effect of polarization on acoustical imaging.

1. INTRODUCTION

So far works on ultrasonic diffraction imaging are based on scalar theory of sound waves. This is not correct as sound has vector nature. When sound propagates in fluids it can be approximated as a scalar wave as there is no polarization. But when sound propagates in solids, its vector nature has to be considered as polarization occurs and both transverse wave and longitudinal wave will occur which travel at different velocities. Vector theory is especially needed when the obstacle size is smaller than the wavelength. This will produce better information on the objects details and better image resolution. Vector theory also provides a study on the effect of polarization on acoustical imaging.

2. DERIVATION OF THE VECTOR ACOUSTIC WAVE EQUATION FOR SOLIDS

We will start with the vectorial Kirchhoff approximation for diffraction and scattering.

The acoustic stress field $\vec{T}$ is analogous to the electric field $\vec{E}$ in the electromagnetic theory. We begin with the solution for the scalar wave function ψ:

$$\psi(\vec{x}) = \oint_S \left[\psi(\vec{x'})\vec{n'} \cdot \nabla' G(\vec{x}, \vec{x'}) - G(\vec{x}, \vec{x'})\vec{n'} \cdot \nabla' \psi(\vec{x'})\right] da' \tag{1}$$

where $\vec{(n')}$ is an inwardly directed normal to the surface S. We then use (1) for each rectangular component of $\vec{E}$ and write the obvious vectorial equivalent:

$$\vec{E}(\vec{x}) = \oint_S \left[\vec{E}(\vec{n'} \cdot \nabla' G) - G(\vec{n'} \cdot \nabla')\vec{E}\right] da' \tag{2}$$

provided the point $\vec{x}$ is inside the volume V bounded by the surface S.

(2) can be rewritten in this form:

$$\mathcal{O} = \oint_S \left[2\vec{E}(\vec{n'} \cdot \nabla' G) - \vec{n'} \cdot \nabla'(G\vec{E})\right] da'$$

The divergence theorem can be used to converty the second term into a volume integral, thus yielding

$$\mathcal{O} = \oint_S 2\vec{E}(\vec{n'} \cdot \nabla' G) da' + \int_V \nabla'^2 (G\vec{E}) d^3x'$$

With the use of $\nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times (\nabla \times \vec{A})$ for any vector field $\vec{A}$, and the vector calculus theorems,

$$\begin{gathered} \int_V \nabla\phi d^3x = \oint_S \vec{n}\phi da \\ \int_V \nabla \times \vec{A} d^3x = \oint_S (\vec{n} \times \vec{A}) da \end{gathered} \tag{3}$$

where ϕ and $\vec{A}$ are any well-behaved scalar and vector functions, we can obtain

$$\mathcal{O} = \oint_S \left[2\vec{E}(\vec{n'} \cdot \nabla' G) - \vec{n'} \cdot \nabla'(G\vec{E}) + \vec{n'} \times (\nabla' \times (G\vec{E}))\right] da' \tag{4}$$

Carrying out the indicated differentiation of the product $G\vec{E}$, and making use of the Maxwell equations, $\nabla' \cdot \vec{E} = 0$, $\nabla' \times \vec{E} = i\omega\vec{B}$, we find

$$\mathcal{O} = \oint_S \left[i\omega(\vec{n'} \times \vec{B})G + 2\vec{E}(\vec{n'} \cdot \nabla' G) - \vec{n'} \cdot \nabla'(G\vec{E}) + \vec{n'} \times (\nabla' \times (G\vec{E}))\right] da'$$

Expansion of the triple cross product, and a rearrangement of terms yields this result,

$$\vec{E}(x) = \oint_S \left[i\omega(\vec{n'} \times \vec{B})G + (\vec{n'} \times \vec{E}) \times \nabla' G + (\vec{n'} \cdot \vec{E})\nabla' G \right] da' \tag{5}$$

Equation (5) is the vectorial equivalent of the scalar formulation (1) with

$$G \longrightarrow \frac{e^{ikr'}}{4\pi r'} e^{iK\vec{n'}\cdot\vec{x}}$$

and its gradient by

$$\nabla' G \longrightarrow -iK\vec{n'}G$$

The following vector Kirchhoff integral relation can be obtained:

$$\vec{E}(x) = \oint_S \left[i\omega(\vec{n'} \times \vec{B})G + (\vec{n'} \times \vec{E}) \times \nabla' G + (\vec{n'} \cdot \vec{E})\nabla' G \right] da' \tag{6}$$

For the acoustic fields, $\vec{E}$ is equivalent to $\vec{T}$ and $\vec{H}$ is equivalent to $\vec{V}$, the velocity field and

$$\vec{T}(x) = \oint_S \left[i\omega(\vec{n'} \times \vec{V})G + (\vec{n'} \times \vec{T}) \times \nabla' G + (\vec{n'} \cdot \vec{T})\nabla' G \right] da' \tag{7}$$

3. PRACTICAL EXAMPLE OF THE APPLICATION OF THE VECTOR DIFFRACTION FORMULA

The generalized Kirchhoff integral for Neumann boundary condition is

$$\psi(\vec{x}) = -\int_{S_1} \frac{\partial \psi}{\partial n'}(\vec{x'}) G_N(\vec{x}, \vec{x'}) da' \tag{8}$$

The vector relation for(8) is

$$\vec{E}_{diff}(\vec{x}) = \frac{1}{2\pi} \nabla \times \int_{apertures} (n \times \vec{E}) \frac{e^{iKR}}{R} da' \tag{9}$$

or

$$\vec{T}_{diff}(\vec{x}) = \frac{1}{2\pi} \nabla \times \int_{apertures} (n \times \vec{T}) \frac{e^{iKR}}{R} da' \tag{10}$$

for the acoustic case. The first example is diffraction by a circular aperture [1]. We consider Fraunhofer diffraction when the observation point is far from the diffracting system, eqn. (10) reduces in this limit to

$$\vec{T}(\vec{x}) = \frac{ie^{iKr}}{2\pi r} \vec{K'} \times \int_{S_1} \vec{n} \times \vec{T}(\vec{x'}) e^{-iKR} da' \tag{11}$$

We consider a plane wave incident at an angle α on a thin, perfectly elastic screen with a circular hole of radius a in it. The polarization vector of the incident wave lies in the plane of incidence.

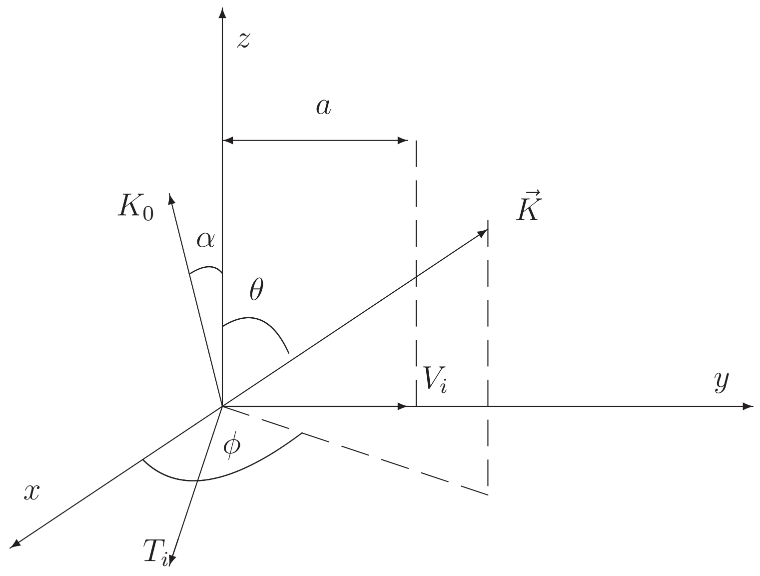

Figure 1. Diffraction by a cicular hole of radius a.

Fig. 1 shows an appropriate system of coordinates. The screen lies in the x-y plane with the opening centered at the origin. The wave is incident from below, so that the domain $z > 0$ is the region of diffraction fields. The plane of incidence is taken to be the x-z plane. The incident wave's stress field, written out explicitly in rectangular components, is

$$\vec{T_i} = \vec{T_0}(S_1 \cos\alpha - S_3 \sin\alpha)e^{-iK(z\cos\alpha + x\sin\alpha)} \tag{12}$$

where $S =$ stiffness.

In calculating the diffraction field with (11) we will make the customary approximation that the exact field in the surface integral may be replaced by the incident field. For the vector relation (11), we need

$$(\vec{n} \times \vec{T_i})_{z=0} = T_0 \vec{s}_2 \cos\alpha \; e^{iK \sin\alpha x'} \tag{13}$$

Then introducing plane polar coordinates for the integration over the opening, we have

$$\vec{T}(\vec{x}) = \frac{ie^{iKrT_0 \cos\alpha}}{2\pi r}(\vec{K} \times \vec{S_2}) \int_0^a \rho d\rho \int_0^{2\pi} d\beta e^{ik\rho[\sin\alpha\cos\beta - \sin\theta\cos(\alpha-\beta)]} \tag{14}$$

where θ, ϕ are the spherical angles of $\vec{k}$. If we define the angular function,

$$\xi = (\sin^2\theta + \sin^2\alpha - 2\sin\theta\sin\alpha\cos\phi)^{1/2}$$

the angular integral can be transformed into

$$\frac{1}{2\pi}\int_0^{2\pi} d\beta' e^{-iK\rho\xi\cos\beta'} = J_0(K\rho\xi)$$

Then the radial integral (14) can be done directly. The resulting stress field is

$$\vec{T}(\vec{x}) = \frac{ie^{iKr}}{r} a^2 T_0 \cos\alpha (\vec{K} \times \vec{S}_2) \frac{J_1(Ka\xi)}{Ka\xi} \tag{15}$$

The time averaged diffracted power per unit solid angle is

$$\frac{dP}{d\Omega} = P_i \cos\alpha \frac{(Ka)^2}{4\pi} (\cos^2\theta + \cos^2\phi \sin^2\theta) \left| \frac{2J_1(Ka\xi)}{Ka\xi} \right|^2 \tag{16}$$

where

$$P_i = \left(\frac{T_0^2}{2z_0} \right) \pi a^2 \cos\alpha, \tag{17}$$

is the total power normally incident on the aperture.

The power radiated per unit solid angle in the scalar Kirchhoff approximation is

$$\frac{dP}{d\Omega} = P_i \frac{(Ka)^2}{4\pi} \cos\alpha \left(\frac{\cos\alpha + \cos\theta}{2\cos\alpha} \right)^2 \left| \frac{2J_1(Ka\xi)}{Ka\xi} \right|^2 \tag{18}$$

If we compare the vector result (16) with (18), we find similarities and differences. Both formulae contain the same “diffraction" distribution factor $|J_1(Ka\xi)/Ka\xi|^2$ and the same dependence on wave number. But the scalar result has no azimuthal dependence (apart from that contained in ξ), whereas the vector expression does. The azimuthal variation comes from the polarization properties of the field, and must be absent in a scalar approximation. For normal incidence ($\alpha = 0$) and $Ka \gg 1$, the polarization dependence is unimportant. The diffraction is confined to a very small angles in the forwarded direction. Then all scalar and vector approximations reduce to the common expression

$$\frac{dP}{d\Omega} = P_i \frac{(Ka)^2}{\pi} \left| \frac{J_1(Ka\sin\theta)}{Ka\sin\theta} \right|^2 \tag{19}$$

4. DERIVATION OF IMAGE FORMATION THEORY

Sofar the ultrasonic image formation is based on the Kirchhoff scalar diffraction theory. We use the angular spectrum approach[2]. This method is different from Kirchhoff theory. It resembles the theory of linear time-invariant filters. Here the complex field distribution across any plane is Fourier analyzed, the various spatial Fourier components can be identified as plane waves travelling in different directions. The field amplitude of any other point can be calculated by adding the contributions of these plane waves, taking due account of the phase shifts they have undergone in propagating to the point in question.

Let the complex field across that plane be represented by $\vec{T}(x, y, 0)$; our ultimate objective is to calculate the consequent field $\vec{T}(x, y, z)$ that appears at a second point P_0 with coordinate (x, y, z).

Across the xy plane, the function $\vec{T}$ has a two-dimensional Fourier transform given by

$$\vec{A}_0(f_x, f_y) = \iint_{-\infty}^{\infty} T(x, y, 0) exp\left[-j2\pi(f_x x + f_y y)\right] dxdy \tag{20}$$

The operation of a Fourier transformation may be regarded as a decomposition of a complicated function into a collection of more simple complex-exponential functions. To emphasize this point of view $\vec{T}$ can be written as an inverse transform of its spectrum

$$\vec{T}(x,y,0) = \iint_{-\infty}^{\infty} A_0(f_x,f_y) exp\left[j2\pi(f_x x + f_y y)\right] df_x df_y \tag{21}$$

Considering that the equation for a unit-amplitude plane wave propagating with direction cosines (α,β,γ) is simply

$$B(x,y,z) = exp\left[j\frac{2\pi}{\lambda}(\alpha x + \beta y + \gamma z)\right]$$

where $\gamma = \sqrt{1-\alpha^2-\beta^2}$.

Thus across the plane $z = 0$, a complex-exponential function $exp[j2\pi(f_x x + f_y y)]$ may be regarded as a plane wave propagating with direction cosines

$$\alpha = \lambda f_x \quad \beta = \lambda f_y \quad \gamma = \sqrt{1-(\lambda f_x)^2-(\lambda f_y)^2} \tag{22}$$

The complex amplitude of that plane-wave component is simply $A_0(f_x,f_y)df_x df_y$, evaluated at ($f_x = \alpha/\lambda$, $f_y = \beta/\lambda$). For this reason, the function

$$\vec{A}_0\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right) = \iint_{-\infty}^{\infty} \vec{T}(x,y,0) exp\left[-j2\pi\left(\frac{\alpha}{\lambda}x + \frac{\beta}{\lambda}y\right)\right] dxdy \tag{23}$$

is called the angular spectrum of the disturbance (23) $\vec{T}(x,y,0)$.

Consider now the angular spectrum of the disturbance $\vec{T}$ across a plane parallel to the xy plane but at a distance z from it. Let the function $\vec{A}(\alpha/\lambda,\beta/\lambda,z)$ represent the angular spectrum $\vec{T}(x,y,z)$ that is,

$$\vec{A}\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},z\right) = \iint_{-\infty}^{\infty} \vec{T}(x,y,z) exp\left[-j2\pi\left(\frac{\alpha}{\lambda}x + \frac{\beta}{\lambda}y\right)\right] dxdy \tag{24}$$

Now if the relation between $\vec{A}_0(\alpha/\lambda,\beta/\lambda)$ and $\vec{A}(\alpha/\lambda,\beta/\lambda,z)$ can be found, then the effects of wave propagation on the angular spectrum of the disturbance will be clear.

To find the desired relation, note that $\vec{T}$ can be written as

$$\vec{T}\left(x,y,z\right) = \iint_{-\infty}^{\infty} \vec{A}\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},z\right) exp\left[j2\pi\left(\frac{\alpha}{\lambda}x + \frac{\beta}{\lambda}y\right)\right] d\frac{\alpha}{\lambda}d\frac{\beta}{\lambda} \tag{25}$$

The $\vec{T}(\vec{x})$ determined by vectorial approach is given by eqn. (15). We now extend $\vec{T}(x)$ to $\vec{T}(x,y,z)$ to take account of the effect of propagation.

It is also to be noted that the vectorial approach is the proper approach for treating the existence of evanescent waves in the angular spectrum. These wave components are strongly attenuated by the propagation phenomenon and so are non-propagating.

5. DISCUSSION ON THE EFFECT OF POLARIZATION ON ACOUSTICAL IMAGING

The effect of polarization on acoustical imaging is seldom discussed. It is often ignored. We have seen from Eqn. (16) the azimuthal variation due to the polarization properties of the acoustic field. For normal incidence, $\alpha = 0$ and $Ka \gg 1$, the polarization dependence is unimportant, the diffraction is confined to very small angles in the forward direction and all scalar and vector approximations reduce to the same expression. For $Ka = \pi$, there is a considerable disagreement between the vector and scalar approximations. It is to be noted that scalar approximation only gives information on the diffracted acoustic field only in a particular component direction if the rectangular coordinates is used.

6. CONCLUSION

So far we have followed the Kirchhoff vector approximation in our work. It would be very useful to follow the exact calculations. There is reason to believe that the vector Kirchhoff result is close to the exact theory, even through the approximation breaks down seriously for $Ka \leq 1$. The vector approximation and exact calculation[3] for a rectangular opening yield results in good agreement even down to $Ka \sim 1$.

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