

FAST BOUNDARY ELEMENT MODELS FOR FAR FIELD PRESSURE PREDICTION

Rick C. Morgans(1), Anthony C. Zander(1), Colin H. Hansen(1), and David J. Murphy(2)

(1) Active Noise and Vibration Control group, School of Mechanical Engineering, University of Adelaide, Australia

(2) Henley Beach, South Australia, Australia

Abstract

The prediction of the pressure field at a distance from an arbitrary structure is often of interest in the acoustic design of products, or for the prediction of the sound field scattered by an incoming plane wave. Analytical techniques are restricted to structures that conform to separable coordinate systems and alternative approaches such as finite or boundary element methods are often used to overcome this limitation. This paper compares results obtained from the analytical solution of a vibrating cap mounted on the surface of a sphere with two alternative boundary element based techniques, a traditional direct boundary element method and a new source superposition technique. The accuracy of the far field pressure solution for both techniques is examined. It has been found that accurate solutions for beamwidth can be obtained when the mesh density is reduced below 6 elements per wavelength.

Nomenclature

$p(x)$	pressure
k	wavenumber
ω	circular frequency
f	frequency
c	speed of sound
n	normal direction
ρ	density
$v_n(x)$	normal velocity
r	radial coordinate
θ	angular coordinate
a	sphere radius
$P_n(\cos \theta)$	Legendre Polynomial
$h_n^{(2)}(r)$	spherical Hankel function of the second kind
u_0	cap velocity
θ_0	cap angle
x_s	position of the source
x	position of the field point
R	distance between x_s and x
$c(x)$	position dependent constant
α_m	constant for source superposition technique
β_m	constant for source superposition technique

Introduction

The prediction of the sound field away from an arbitrary structure is of interest in many fields of acoustics. Applications in underwater sound include the design of transducers that radiate sound with desired directivity characteristics [1] as well as in sonar systems, where the sound field generated by reflections (scattering) of an incident plane wave is used to detect

the shape of the submerged hard structure [2, Chapter 10].

Traditional approaches to predicting radiation or scattering from structures have been limited to either analytical solutions of the governing equations, or to high or low frequency approximations to these equations [2, 3]. The analytical solutions are limited to structures with surfaces that conform to constant coordinate values in a small number of separable co-ordinate systems [4], such as cylinders or spheres. As most structures are not of this form, and the frequencies of interest lie between the low and high frequency approximations, alternative approaches must be sought.

Numerical techniques such as Finite Element Analysis (FEA) [5, 6] or the Boundary Element Method (BEM) [7, 8] have been used to predict sound fields from arbitrary structures. However, while these methods can eliminate problems associated with analytical techniques, it has been found that fully 3-D FEA can become intractable for large models and high frequencies, and unsuitable for application to optimisation techniques [9]. There is also evidence that fully 3-D direct BEM is similarly unsuitable for the mid to high frequencies [7].

This paper investigates the application of a fully 3-D direct BEM [8], as well as a relatively new source superposition BEM [10], to the modelling of a vibrating cap mounted on the surface of a sphere. Analytical solutions to this geometry are readily available [3]. First, the theoretical backgrounds to the analytical solution and both the direct BEM and the source superposition method are given, followed by an explanation of the figure of merit used to compare solutions (the beamwidth). The exact analytical results are compared with the results obtained using the numerical techniques, and an analysis of the speed of solution is made. The accuracy of the far field pressure solution for both techniques is examined when the mesh density is reduced to below 6 elements per wavelength. Finally, conclusions are drawn as to the

utility of numerical modelling using boundary element methods.

Theory

The governing equation of time harmonic linear acoustics is the scalar Helmholtz equation [8, 9, 12, 13]:

$$\nabla^2 p(x) + k^2 p(x) = 0 \quad (1)$$

where $p(x)$ is the pressure, $k = \omega/c$ is the wavenumber, $\omega = 2\pi f$ is the circular frequency, f is the frequency and c is the speed of sound in the medium, in this case air. This equation is derived from the linearised equations of conservation of momentum and mass. To be solved, Equation (1) requires appropriate boundary conditions. The velocity at the interface between a solid and a fluid can be related to the gradient of pressure as:

$$\frac{\partial p(x)}{\partial n} = i\rho\omega v_n(x) \quad (2)$$

where n is the normal direction, ρ is the density of the fluid and $v_n(x)$ is the normal velocity. For external problems, where the sound radiates away from the structure to infinity, another boundary condition called the Sommerfield radiation condition is needed to make sure only outgoing waves are present.

Analytical solutions to Equation (1) exist for separable coordinate systems [3]. For the special case of a vibrating cap mounted on the surface of a sphere (Figure 1), with azimuthally symmetric loading, the pressure at any point outside the sphere can be calculated by

$$p(r, \theta) = \frac{\rho c}{i} \sum_{n=0}^{\infty} U_n P_n(\cos \theta) \frac{h_n^{(2)}(kr)}{h_n^{(2)}(ka)} \quad (3)$$

where r and θ are the radial and angular coordinates, a is the radius of the sphere, $P_n(\cos \theta)$ is a Legendre Polynomial, $h_n^{(2)}(r)$ is a spherical Hankel function of the second kind, a prime means differentiation with respect to r and

$$U_n = \frac{1}{2} u_0 [P_{n-1}(\cos \theta_0) - P_{n+1}(\cos \theta_0)] \quad (4)$$

for a spherical cap on the surface of a sphere vibrating with velocity u_0 and subtending angle θ_0 . A MATLAB program has been written to calculate Equation (3).

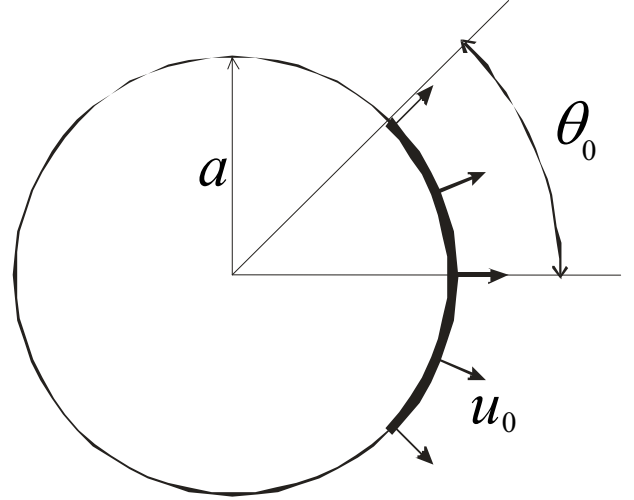


Figure 1. A cap subtending angle θ_0 mounted on the surface of a sphere of radius a , vibrating with velocity u_0 .

No analytical solutions exist for arbitrary geometries, and alternative numerical methods are often used. One approach to solving Equation (1) numerically would be to discretise it directly and solve for the pressure at every point in the field. This is the approach that FEA takes, but there are limitations when solving problems in an infinite domain, which must be truncated in order to solve the problem. The Sommerfield radiation condition must be enforced, otherwise reflections from the boundary can affect the result. The development of appropriate boundary conditions and their incorporation into a finite element analysis is a topic of ongoing research [11].

Another approach is to replace the solid surface that is being modelled with a distribution of fundamental solutions to Equation (1). A monopole is a fundamental solution that can be derived from the linearised equations of conservation of momentum and mass with the addition of a localised volume velocity injection. This represents the sound field due to a point source, and is called the “free space” Green’s function:

$$g(x_s | x) = \frac{e^{ikR}}{4\pi R} \quad (4)$$

where R is the distance between x_s , the position of the source, and x , the position of the field point. Note that Equation (4) is singular when the source and field point coincide.

A dipole is also a fundamental solution of Equation (1), derived from the linearised equations of conservation of momentum and mass with the addition of a localised force. It represents the sound field of two monopoles in close proximity operating 180° out of phase and is the directional derivative of Equation (4):

$$\frac{\partial g(x_s | x)}{\partial n} \quad (5)$$

Conceptually, any solid surface can be replaced by a distribution of monopoles and dipoles. The effect of the surface is replaced by the action of a distribution of forces aligned normal to the boundary, and the imposed velocity is replaced with the injection of volume velocity. Figure 2 shows a representation of this effect.

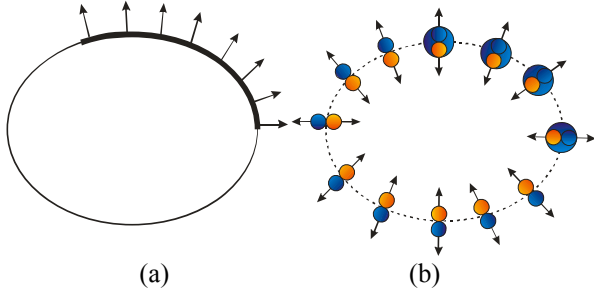


Figure 2. A solid surface with an imposed velocity over part of the surface, (a), can be replaced by a suitable distribution of monopoles and dipoles, (b).

Direct BEM

The Kirchoff-Helmholtz (K-H) integral equation [3, 8, 10, 12]:

$$c(x)p(x) = - \int_s i\rho\omega v_n(x_s)g(x_s|x) + p(x_s)\frac{\partial g(x_s|x)}{\partial n} ds \quad (6)$$

where $c(x)$ is a position dependent constant, can be derived from either physical arguments using monopoles and dipoles [13] or from vector calculus and Green's theorem [10,13]. This is the fundamental equation of direct BEM, and shows that the pressure at any point can be represented by the surface integral of a combination of monopoles and dipoles. In this equation, the dipole source strength is weighted by the surface pressure. Given a distribution of surface normal velocity, once the surface pressure is found, any pressure field can be calculated.

The direct BEM finds the surface pressure by discretising Equation (6) with n_n nodes and n_e elements similar to those used in FEA. If the field point is positioned at each surface node (or "collocated") then a series of n_n equations for the n_n surface pressures can be found for a given velocity distribution. The equations are generated by numerical integration over each element, and the integration technique used must be capable of dealing with the singularities found at the locations of the monopoles and dipoles. The equations can be formed into a matrix and inverted using standard linear algebra techniques. Once the matrix is inverted, and the surface pressures known, the field pressures can be calculated.

There are a number of disadvantages to the direct BEM approach. The K-H integral equation represents the sound field on the exterior of a finite volume. At the natural frequencies of the interior of the finite volume,

the exterior problem breaks down and the matrix becomes ill-conditioned. This is well documented [14] and many solutions have been attempted [15, 16].

Another problem occurs when the two surfaces of interest are brought close together, resulting in "thin-shape breakdown" [17]. This means that although some geometries are probably best represented with a thin surface, a direct BEM simulation will have to assume the geometry is contained in an enclosing volume to avoid thin shape breakdown, leading to an increased number of nodes and hence solution time.

The direct BEM code used in this research is HELM 3D [8], a Fortran 77 implementation using linear elements. The CHIEF method is used to overcome the interior natural frequency problem. This technique solves an overdetermined system of equations formed placing extra points (x) inside the volume of interest. Provided the points are not placed at a nodal line of the interior solution, this will improve the matrix condition number and allow the matrix to be solved in a least squared sense. For this application the code was modified to accept quarter symmetric models, a change necessary to reduce overall run time.

Source superposition

The source superposition technique of Koopmann and Fahline [10] does not solve the K-H equation directly. Instead, it uses an expansion of the pressure at a field point in terms of a series of monopoles and dipoles, each placed at the centroid of each element of the discretised surface:

$$p(x) = \sum_{m=1}^{n_e} s_m \left\{ \alpha_m g(x_m|x) + \beta_m \frac{\partial g(x_m|x)}{\partial n_m} \right\} \quad (7)$$

where n_e is the number of elements, s_m is the source strength, α_m and β_m are constants depending on whether the source is a monopole, dipole or combination of the two (tripole). Monopoles are used to represent sources on the surface of an infinite baffle, dipoles are used to represent thin surfaces and tripoles to represent the surface of the exterior of a finite volume. The use of tripoles eliminates the interior natural frequency problem of the direct BEM and this technique is capable of modelling thin surfaces directly.

The normal velocity can be found using Equations (2) and (7):

$$v_n(x) = \frac{1}{ik\rho c} \sum_{m=1}^{n_e} s_m \left\{ \alpha_m \frac{\partial g(x_m|x)}{\partial n} + \beta_m \frac{\partial^2 g(x_m|x)}{\partial n \partial n_m} \right\} \quad (8)$$

and the volume velocity over element μ of the boundary surface can be found by integrating Equation (8) over the element surface,

$$\begin{aligned}
U_\mu(x) &= \int_s v_n(x) ds \\
&= \int_s \frac{1}{ik\rho c} \sum_{m=1}^n s_m \left\{ \alpha_m \frac{\partial g(x_m|x)}{\partial n} + \beta_m \frac{\partial^2 g(x_m|x)}{\partial n \partial n_m} \right\} ds
\end{aligned} \quad (9)$$

for $\mu=1,\dots,n_e$. This produces a series of n_e equations with n_e unknown source strengths. The resulting matrix can be inverted to find the source strengths, s_m . Once these strengths are found, the sound field can be reconstructed using Equation (6).

The source superposition code used in this research is the Fortran 77 program Power [9]. This program has also been modified for quarter symmetry.

Analytical solution

The analytical solution for a 45° spherical cap mounted on the surface of a unit sphere has been calculated using Equations (3) and (4). The infinite sum in Equation (3) was truncated at 100 terms, and the pressure calculated in the far field. Figure (3) shows a polar plot of the magnitude of the measured pressure, normalised by the maximum pressure, for a 45° spherical cap, for three different non-dimensional frequencies ($ka = 3, 10, 20$).

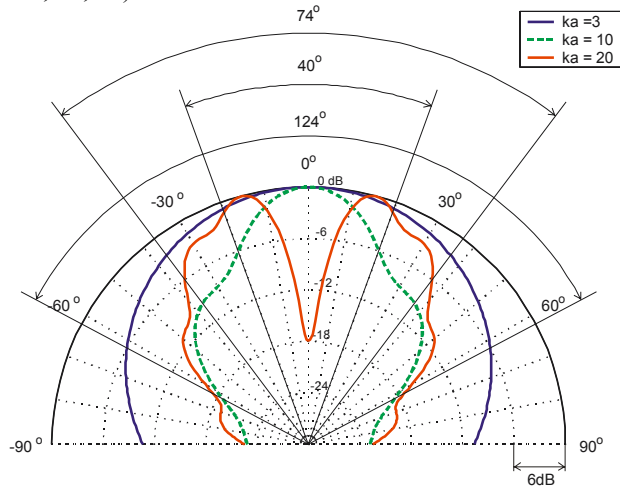


Figure 3. Polar plot of the magnitude of the measured pressure, normalised by the maximum pressure, for a 45° spherical cap mounted on the surface of a sphere.

Figure 3 also shows the beamwidth for each of these frequencies. The beamwidth is defined as the angle formed by the -6 dB points, with reference to the maximum reading, and the source centre [18] and is a measure of the distribution of sound in the specified plane. Figure 4 shows a plot of the beamwidth versus frequency, and is the baseline for comparison with the numerical methods.

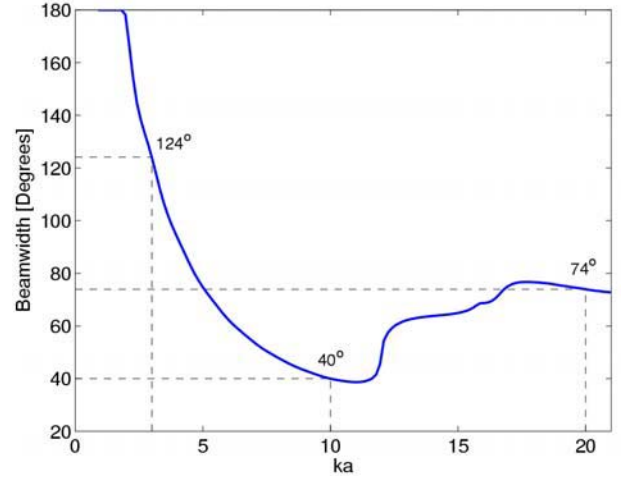


Figure 4. The variation of beamwidth with frequency for a 45° spherical cap mounted on the surface of a sphere.

Results

Simulations of a 45° spherical cap mounted on the surface of a unit sphere have been undertaken for both the direct BEM and source superposition techniques. Figure 5 shows the surface mesh used to discretise the sphere, at a nominal 6 elements per wavelength. Note the quarter symmetry. A unit normal velocity was placed over the vibrating cap, represented by the darker area in Figure 5. The pressure was calculated at a radius of $18a$. In this case the number of variables to be solved for the direct BEM is 1476 compared to 1412 for the source superposition technique.

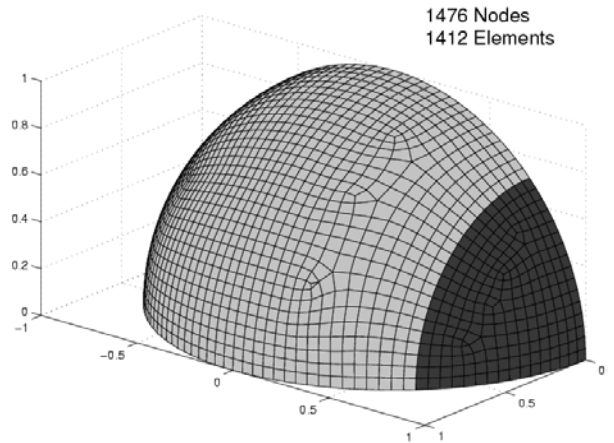


Figure 5. Surface mesh of the 45° spherical cap mounted on the surface of a sphere.

The beamwidth of the sphere was calculated for 135 non-dimensional frequencies ka ranging linearly from 1 to 21. The upper frequency was chosen to limit the run time required for the direct BEM method. Figure 6 shows a comparison of the error, ε , between the analytical and numerical results, defined as,

$$\varepsilon = \frac{|bw_n - bw_a|}{bw_a} \quad (7)$$

where bw_n and bw_a are the numerical and analytical beamwidth respectively, for both direct BEM and the source superposition method. The agreement between both methods and the analytical solution is excellent, with errors less than 1% for the direct BEM. The source superposition technique produces a larger error of about 8% at a ka of 12. The source superposition technique was found to produce results ~3 times faster than the direct BEM. On an Intel P4 1500 MHz, running Windows XP, the run time is 379 seconds per frequency for the direct BEM and 115 seconds per frequency for the source superposition technique. For comparison the analytical technique takes ~1.5 seconds per frequency.

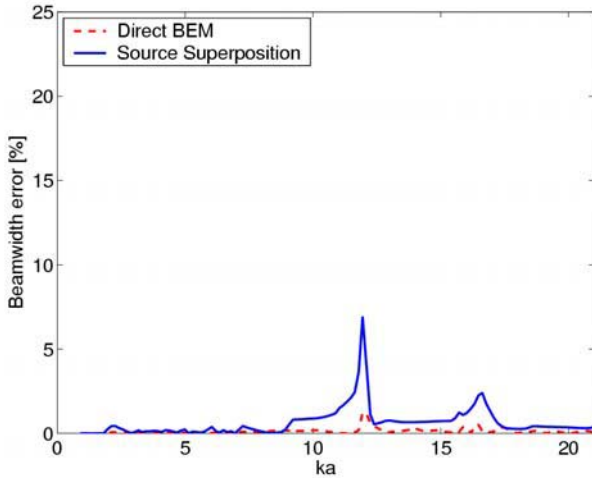


Figure 6. Error in the beamwidth with mesh density 6 elements per wavelength.

The mesh density was chosen using the standard finite element rule of thumb of 6 linear elements per wavelength [19]. Figure 7 shows a plot of a mesh with nominally 3 elements per wavelength. The node and element count is reduced by at least a factor of 3, with a corresponding reduction in run time to 36 seconds per frequency for the direct BEM and 6 seconds per frequency for the source superposition technique. The source superposition technique is now 6 times faster than the direct BEM.

The time taken to calculate the solution has been reduced by a factor of 10 for the direct BEM and a factor of 20 for the source superposition technique. With this massive reduction in computation time, accurate solutions, at least in the far field, are possible.

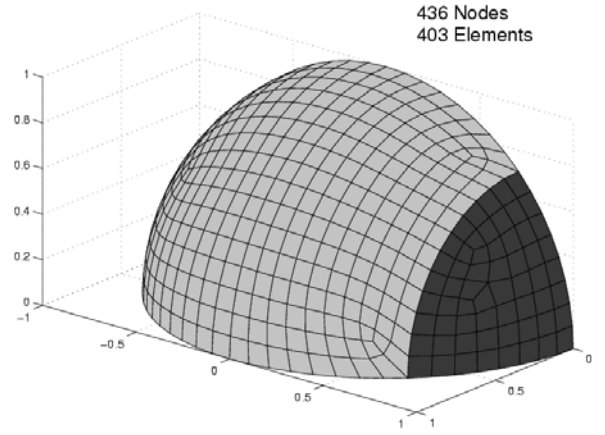


Figure 7. Surface mesh of the 45° spherical cap mounted on the surface of a sphere.

Figure 8 shows the errors in the beamwidth for the mesh shown in Figure 7. Again the agreement between the two methods and the analytical solution is excellent, with the error in the direct BEM less than 1% for most of the bandwidth. The source superposition error is less than 5% for most of the frequency range, except for a ka of 12, where it jumps to 25%. Equation (7) is a very sensitive measure of the sound field where the rate of change of beamwidth with frequency is large, because small changes in the beamwidth will lead to large changes in the error estimate. This quantity is perhaps not a good measure of error under these conditions. Figure (9) shows a comparison of the beamwidth of the analytical solution and the source superposition method with 3 elements per wavelength. The large errors at a ka of 12 are associated with the sharp jump in beamwidth, and this level of error predicted by Equation (7) is deemed acceptable for most purposes.

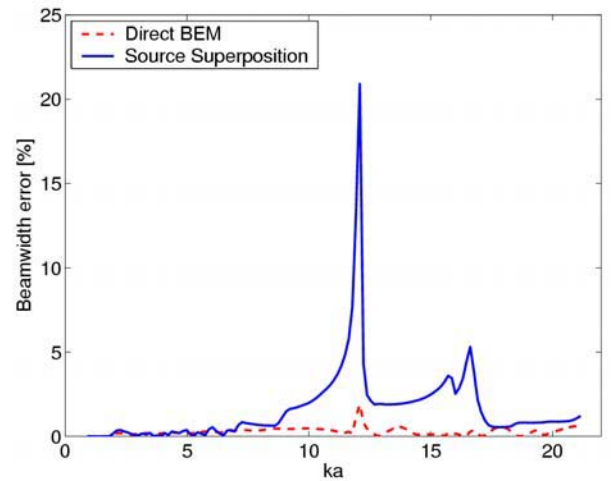


Figure 8. Error in the beamwidth with mesh density 3 elements per wavelength.

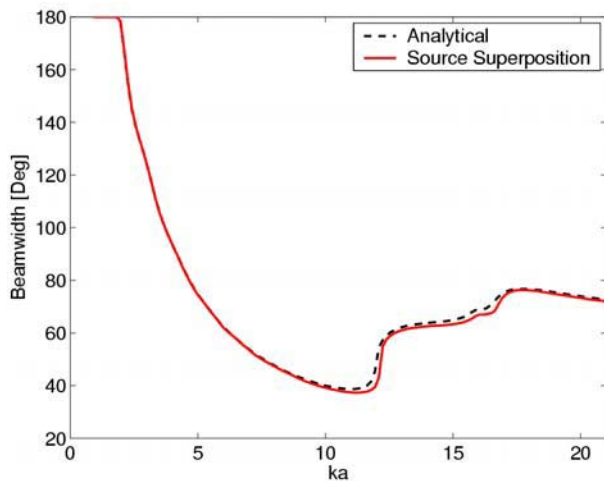


Figure 9. Beamwidth for analytical solution and source superposition 3 elements per wavelength.

Conclusions

Boundary element numerical models of a 45° spherical cap mounted on the surface of a sphere have been developed, which accurately model the beamwidth over the frequency range simulated. The source superposition technique is found to give similar results to the direct BEM, but is 3 to 6 times faster, with a minor loss of accuracy.

A major finding of this work is that far field solutions that calculate beamwidth do not need as high a mesh density as traditionally associated with the BEM, reducing calculation time dramatically without compromising accuracy.

This work can be regarded as verification [20], ensuring that the correct equations are solved by the software. The next stage of the work, validation, that is comparison to experiments to see how well the equations represent the physical system, is yet to be undertaken for the source superposition technique and beamwidth. It should be noted, however, that the source superposition technique is well validated for calculations of radiated acoustic power [10], and that accurate calculations of power imply that the far field is correctly modelled. The source superposition technique is expected to perform well when calculating beamwidth for models other than a simple sphere.

References

- [1] Van Buren, A. L., Luker, L. D., Jevnager, M. D. and Tims, A. C., "Experimental constant beamwidth transducer", *Journal of the Acoustical Society of America*, 73(6):2200-2208, 1983.
- [2] Junger, M. C. and Feit, D., *Sound, Structures, and Their Interaction*, Acoustical Society of America, 1993.
- [3] Morse, P. M. and Ingard, K. U., *Theoretical acoustics*, Princeton University Press, 1986.
- [4] Morse, P. M. and Feshbach, H., *Methods of Theoretical Physics*, pt. 1, McGraw-Hill, New York, 1953.
- [5] ANSYS 5.7 Theory Manual, 12th Edition, Ansys Inc, Canonsburg, PA, 2001.
- [6] Dey, S., Shirron, J. J. and Couchman, L. S., "Mid-frequency structural acoustic and vibration analysis in arbitrary, curved three-dimensional domains", *Computers and Structures*, 79:617-629, 2001.
- [7] von Estorff, O., *Boundary Elements in Acoustics*, WIT Press, 2000.
- [8] Wu, T. W., *Boundary Element Acoustics: Fundamentals and Computer Codes*, WIT Press, 2000.
- [9] Morgans, R. C., Zander, A. C., Hansen, C. H. Murphy, D. J., "Distortion and directivity modelling of horn loaded loudspeakers", C00107712, Strategic Partnerships with Industry - Research and Training Scheme Application Form for Year 2001 Grants, 2000.
- [10] Koopmann, G. H. and Fahline, J. B., *Designing quiet structures: a sound power minimization approach*, Academic Press, 1997.
- [11] Gerdes, K., "A review of infinite element methods for exterior Helmholtz problems", *Journal of Computational Acoustics*, 8(1):43-61, 2000.
- [12] Pierce, A.D., *Acoustics: an introduction to its physical principles and applications*, Acoustical Society of America, New York, 1994.
- [13] Fahy, F., *Foundations of Engineering Acoustics*, Academic Press, 2001.
- [14] Copley, L. G., "Fundamental results concerning integral representations in acoustic radiation", *Journal of the Acoustical Society of America*, 44(1):28-32, 1968.
- [15] Schenck, H. A., "Improved integral formulation for acoustic radiation problems", *The Journal of the Acoustical Society of America*, 44:41-58, 1968.
- [16] Burton, A. J. and Miller, G. F., "The application of integral equation methods to the numerical solutions of some exterior boundary-value problems", *Proceedings of the Royal Society of London A*, 323:201-210, 1971.
- [17] Martinez, R., "The thin-shape breakdown (TSB) of the Helmholtz integral equation", *Journal of the Acoustical Society of America*, 90(5):2728-2738, 1991.
- [18] Davis, D. and Davis, C., *Sound system engineering*, Butterworth-Heinemann, 1997.
- [19] Marburg, S., "Six boundary elements per wavelength: is that enough?", *Journal of Computational Acoustics*, 10(1): 25-51, 2002.
- [20] Babuska, I. and Oden, J. T., "Verification and validation in computational engineering and science: basic concepts", *Computer Methods in Applied Mechanical Engineering*, 193:4057-4066, 2004.