Comparison of the non-uniform Kirchhoff and Ffowcs Williams and Hawkings methods of aeroacoustics in application to the sound radiated by a thin plate vibrating in its own plane in a viscous fluid

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ABSTRACT

A thin rectangular acoustically small plate vibrating in its own plane in a viscous fluid is considered. The sound radiated by such vibrations is evaluated by using both non-uniform Kirchhoff and Ffowcs Williams and Hawkings equations. Both non-uniform Kirchhoff and Ffowcs Williams and Hawkings equations include the sound generated by Lighthill's quadrupole sources which are described by a volume integral over the regions in the fluid containing vorticity. For the purpose of evaluating the volume integral, the fluid is separated into three regions: the viscous boundary layer where the fluid motion is predominantly rotational, most of the fluid where its motion is potential, and a narrow transitional region. It is shown that the boundary layer does not generate any sound, whereas the transitional area generates the sound with dipole directivity. As Kirchhoff's integrals over the surface of the plate vanish, it is concluded that all generated sound can be attributed to the Lighthill's volume sources. Ffowcs Williams and Hawkings equation describes, apart from Lighthill's sources, the dipole sound generated by tangential forces acting on the surface of the plate. It is demonstrated that, for the plate considered, the amplitude of this sound is significant, and, therefore, the two methods produce significantly different predictions for the radiated sound. The obtained predictions for the radiated sound are discussed and experimental measurements to verify the obtained results are proposed. Also, recommendations for the practical use of both methods are suggested.

INTRODUCTION

The prediction of sound generated by a fluid flow became an important topic of research with the development and widespread use of jet aircraft in 1950s. The first significant contribution to this topic was made by Sir James Lighthill (1952). He showed that the sound radiated by a turbulent flow without boundaries was controlled by the wave equation with the source term determined by the "Lighthill's stress tensor", which represents all non-acoustic stresses in the fluid. Lighthill also showed that the source term corresponds to quadrupole sound. Lighthill's theory is often called the "acoustic analogy".

Curle (1955) extended Lighthill's theory to a flow with solid boundaries. Curle stated that, for an immoveable boundary, the radiated sound consisted of Lighthill's quadrupole sound as well as the dipole sound originating at the rigid boundary. The amplitude of the dipole sound was determined by the total force acting upon the fluid from the boundary including viscous tangential force.

Ffowcs Williams and Hawkings (1969) extended the theory of the "acoustic analogy" to a flow with moving boundaries. They showed that the motion of the boundaries led to the appearance of a third term in the equation for the radiated sound amplitude. This term is determined by the normal velocity of the boundary with respect to a stationary observer and, therefore, describes the monopole sound. For a stationary boundary, the FW-H equation is reduced to Curle's equation.

Since its derivation, the Ffowcs Williams and Hawkings equation has become the foundation for one of the most frequently used methods of prediction of sound radiated by fluid flow near rigid surfaces. A brief list of applications where the FW-H equation is utilised includes rotating helicopter blades, rotating fans, and flow near an airfoil. This equation is also used in the prediction of noise radiated by moving ships and ship propellers. It is also the foundation of a helicopter noise prediction code, which is employed extensively by the helicopter industry. (See Zinoviev (2007) for a list of references).

Despite being widely used, the FW-H equation appears to lack conclusive experimental verification. Bies and Zinoviev (2007) discussed two early experiments by Clark and Ribner (1969) and by Heller and Widnall (1969), which showed discrepancy with the predictions of the FW-H theory of up to 5 dB. Bies (1992) investigated the noise produced by a circular saw and reported that the measured noise was 2.5 dB lower than predicted.

In view of these discrepancies, Zinoviev and Bies (2004) have conducted a critical analysis of the historically first paper on sound generation by a flow near boundaries (Curle 1955). The authors have shown that Curle's derivation contains erroneous evaluation of a volume integral. They also showed that, if the integral is evaluated correctly, Curle's derivation leads not to Curle's equation, which is the same as the FW-H equation for a stationary boundary, but to a different equation, which includes Lighthill's quadrupole sources as well as surface distributions of dipole and monopole sources as described by Kirchhoff integrals (Stratton 1941). The obtained equation differs from the FW-H equation by the appearance of the terms describing the sources at rigid boundaries. This difference will be discussed below in this paper.

More recent experiments (Eschricht et al 2007, Greschner et al 2007) still continue to show a discrepancy of a few decibels between the predictions of FW-H theory and experimental data.

An attempt to verify Curle's theory has been recently made by Leclercq and Doolan (2009). These authors conducted a detailed analysis of sound radiation by two rigid blocks in turbulent air flow in an anechoic wind tunnel. The analysis involves numerical simulations of the flow around the blocks as well as experimental measurements the radiated sound amplitude together with the force acting upon the downstream block. While the authors show that their experimental results and theoretical predictions based on Curle's theory are within 1 dB of each other, there are some uncertainties in their method that can be pointed out.

Firstly, in their experimental setup, the sound is radiated by both blocks, whereas they measure directly only the force acting upon the downstream block. The force acting upon the upstream block is also required to evaluate the radiated sound, and the authors make an estimate of it based on their numerical results. At the same time, the authors show that their numerical prediction for the force acting upon the downstream block is about 20 dB lower than the experimental measurements. The authors attributed this difference to their use of 2-dimensional formulation in their numerical simulation of the flow. It is clear that direct measurements of the force on the upstream block or using a better method of numerical flow simulation would add significantly to the reliability of the experimental results.

Secondly, it is not clear from the paper (Leclercq and Doolan 2009) whether the authors included viscous forces in their numerical estimates of the total force acting upon the blocks. As shown below in this paper, the force in Curle's formulation includes also the viscous component.

Overall, the analysis by Leclercq and Doolan (2009) can definitely provide the solid foundation for further research. At the same time, their current claim that their results "give increased confidence in the use of Curle's theory" appears to be somewhat premature.

Apart from conducting complex experiments, the two methods can be compared by applying them to a simple case where the solution can be found easily. Zinoviev (2007) applied the FW-H equation to three examples: a stationary object in a variable velocity field; a solid object embedded in a turbulent flow; and a thin plate vibrating in its own plane in a viscous fluid. As further investigation of the vibrating plate showed that it demonstrated more complex behaviour than the one suggested in Zinoviev (2007), it has become necessary to consider this case in more detail.

This paper contains an investigation of sound generation by the thin plate vibrating in its own plane in a viscous fluid. First, the difference in the FW-H and non-uniform Kirchhoff equations are demonstrated. Second, the sound amplitude is calculated by the non-uniform Kirchhoff and FW-H equations. Third, it is shown that the two methods predict significantly different results for the sound amplitude. Fourth, a method for experimental verification of the obtained results is suggested.

NON-UNIFORM KIRCHHOFF AND FW-H EQUATIONS

Non-uniform Kirchhoff equation

Using the fundamental laws of mass and momentum conservation for the motion of a fluid, Lighthill (1952) showed that sound generation and propagation in a turbulent fluid flow without boundaries was determined by the following wave equation with respect to the fluid density, ρ :

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \nabla^2 \rho(\mathbf{x}, t) = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}, \quad i, j = 1, 2, 3 . \tag{1}$$

In Eq. (1), c_0 is the sound speed and T_{ij} is Lightill's stress tensor given by:

$$T_{ij} = \rho u_i u_j + p_{ij} - c_0^2 \rho \delta_{ij}, \tag{2}$$

where $\mathbf{u}$ is the fluid velocity vector, δ_{ij} is Kronecker's delta, and p_{ij} is the compressive stress tensor determined as follows:

$$p_{ij} = p\delta_{ij} + \mu \left\{ -\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} + \frac{2}{3}\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij} \right\}, \tag{3}$$

where p is the pressure and μ is the viscosity of the fluid.

Note that Eq. (2) contains a non-linear term proportional to $u_i u_j$. In this article, the fluid velocity is considered to be much smaller than the speed of sound and, therefore, all such terms are neglected in further analysis.

Eq. (1) is a non-uniform wave equation with respect to the density, ρ. If the harmonic temporal dependence, $\mathrm{e}^{-\mathrm{i}\omega t}$, is assumed, Eq. (1) is reduced to Helmholtz equation, which general solution is well-known (Korn 1971, Eq. 15.6-59).

$$\begin{aligned} &\rho'_K(\mathbf{x}) = \rho(\mathbf{x}) - \rho_0(\mathbf{x}) = \\ &\frac{1}{4\pi}\iiint_V \frac{\partial^2 T_{ij}(\mathbf{y})}{\partial y_i \partial y_j}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}dV + \frac{1}{4\pi}\iint_S \frac{\partial \rho'_K(\mathbf{y})}{\partial n}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}dS - \\ &\frac{1}{4\pi}\iint_S \rho'_K(\mathbf{y})\frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right)dS. \end{aligned} \tag{4}$$

In Eq. (4), ρ_0 is the value at equilibrium, $r = |\mathbf{x} - \mathbf{y}|$, $\mathbf{x} = (x_1, x_2, x_3)$ is the radius-vector of the observation point, $\mathbf{y} = (y_1, y_2, y_3)$ is the radius-vector of the source point, S is any closed surface in the fluid, V is the volume of the fluid outside S where $T_{ij} \neq 0$, and $\mathbf{n}$ is the normal vector to S directed outside V. This direction of the normal is assumed throughout this paper.

The above Eq. (4) is the most general form of the non-uniform Kirchhoff formulation considered in this paper. As this equation is a general solution of Lighthill's non-uniform wave equation (Eq. (1)), which is derived without any significant assumptions about the fluid flow, this equation is also valid for a wide variety of flow types including viscous and non-linear flows.

It has to be noted that Eq. (4) differs from Kirchhoff equation known from literature (Brentner & Farassat 1997). Whereas the traditional Kirchhoff equation implies that all Lighthill's acoustic sources are within the closed surface *S*, Eq. (4) allows Lighthill's sources to be outside this surface. Their contribution to the radiated sound wave is determined by the volume integral in Eq. (4), which is not present in the traditional Kirchhoff equation. To distinguish between Eq. (4) and the traditional Kirchhoff equation, the author has named the former "non-uniform Kirchhoff equation".

For flows with low Mach number, where non-linear terms can be neglected, the pressure and density fluctuations are linked by the following simple equation:

$$p' = \rho' c_0^2. \tag{5}$$

In this case, the non-uniform Kirchhoff equation (Eq. (4)) can be re-written in terms of pressure fluctuations, p':

$$\begin{aligned} p'_K(\mathbf{x}) &= p(\mathbf{x}) - p_0(\mathbf{x}) = \\ &\frac{1}{4\pi}\iiint_V \frac{\partial^2 T_{ij}(\mathbf{y})}{\partial y_i \partial y_j}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}dV + \frac{1}{4\pi}\iint_S \frac{\partial p'_K(\mathbf{y})}{\partial n}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}dS - \\ &\frac{1}{4\pi}\iint_S p'_K(\mathbf{y})\frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right)dS. \end{aligned} \tag{6}$$

If the region of the acoustic sources is acoustically small, it can be shown from the Navier-Stokes equation that

$$\frac{\partial p}{\partial n} = -\rho \frac{\partial u_n^p}{\partial t}, \tag{7}$$

where $\mathbf{u}^p$ is the potential component of the velocity. Taking into account Eq. (7), non-uniform Kirchhoff equation (6) can be rewritten as follows:

$$\begin{aligned} p'_K(\mathbf{x}) &= \frac{1}{4\pi}\iiint_V \frac{\partial^2 T_{ij}(\mathbf{y})}{\partial y_i \partial y_j}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}dV - \\ &\frac{1}{4\pi}\frac{\partial}{\partial t}\iint_S \rho_0 u_n^p(\mathbf{y})\frac{\mathrm{e}^{\mathrm{i}kr}}{r}dS - \frac{1}{4\pi}\iint_S p'_K(\mathbf{y})\frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right)dS. \end{aligned} \tag{8}$$

The volume integral in Eq. (8) represents Lighthill's quadrupole sound, whereas the surface integrals represent the field of all acoustic sources located inside *S* as the field of a layer of acoustic sources on *S*. The second term on the right is the field of a layer of monopole sources with the strength, $\rho_0 \partial u_n^p / \partial t$, and the third term on the right is the field of a layer of dipole sources with the strength, *p'*.

Note that no assumptions about viscosity of the fluid have been made in the derivation of Eq. (8). Therefore, it is valid also for flows where the fluid viscosity cannot be neglected.

Ffowcs Williams and Hawkings equation

Based on the same fundamental conservation laws as Lighthill's theory, but using a different mathematical approach, Ffowcs Williams and Hawkings (1969) derived an equation determining the sound radiated by a fluid flow near moving solid boundaries. If the surface *S* corresponds to an impenetrable solid boundary, the FW-H equation in its differential form can be formulated as follows (Brentner and Farassat 1997):

$$\begin{aligned} &\left[\frac{\partial^2}{\partial t^2} - c_0^2\nabla^2\right]p'(\mathbf{x},t) = \\ &\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} + \frac{\partial}{\partial x_i}\left[p_{ij}n_j\delta(S)\right] - \frac{\partial}{\partial t}\left[\rho_0 v_n \delta(S)\right], \end{aligned} \tag{9}$$

where $\delta(S)$ is the Dirac delta function which is zero everywhere except on *S*, and v_n is the normal component of the velocity of the boundary.

For small Mach numbers, $M = u/c_0 << 1$, the solution of Eq. (9) is provided by Ffowcs Williams and Hawkings (1969), Eq. (5.1). For the harmonic temporal dependence, $e^{-\mathrm{i}\omega t}$, this solution can be written as follows:

$$\begin{aligned} p'_{FW-H}(\mathbf{x}) &= \\ &\frac{1}{4\pi}\iiint_V \frac{\partial^2 T_{ij}(\mathbf{y})}{\partial y_i \partial y_j}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}dV + \frac{1}{4\pi}(\mathrm{i}\,\omega)\iint_S \rho_0 v_n(\mathbf{y})\frac{\mathrm{e}^{\mathrm{i}kr}}{r}dS + \\ &\frac{1}{4\pi}\frac{\partial}{\partial x_i}\iint_S p_{ij}(\mathbf{y})n_j\frac{\mathrm{e}^{\mathrm{i}kr}}{r}dS. \end{aligned} \tag{10}$$

With a transformation of the last term on the right, Eq. (10) takes the following form:

$$\begin{aligned} p'_{FW-H}(\mathbf{x}) &= \\ &\frac{1}{4\pi}\iiint_V \frac{\partial^2 T_{ij}(\mathbf{y})}{\partial y_i \partial y_j}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}dV + \frac{1}{4\pi}(\mathrm{i}\,\omega)\iint_S \rho_0 v_n(\mathbf{y})\frac{\mathrm{e}^{\mathrm{i}kr}}{r}dS - \\ &\frac{1}{4\pi}\iint_S P_i(\mathbf{y})\frac{\partial}{\partial y_i}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right)dS, \end{aligned} \tag{11}$$

where $P_i = p_{ij}n_j$ is the *i*-th component of the force acting upon the boundary from the fluid.

Note that the differentiation in the volume integral in Eq. (11) is done under the integral sign, as the possibility to exchange the order of integration and differentiation in such equations is stated by both Lighthill (1952) and Ffowcs Williams and Hawkings (1969).

Comparison of the two equations

While the non-uniform Kirchhoff equation (8) and the FW-H equation (11) are derived based on the same conservation laws, these two equations are different in two important aspects.

First, the monopole term (the second term on the right) in Eq. (8) contains the *potential* component of the fluid velocity on the boundary, whereas the monopole term in Eq. (11) contains the *total* fluid velocity that may include also the *rotational* component of the velocity, which is related to vorticity. However, any investigation of the effect of this difference on prediction of the radiated sound is outside the scope of this work.

Second, the dipole term (the third term on the right) in Eq. (8) contains only *pressure*, whereas the dipole term in Eq. (11) contains the *total force* on the boundary including the *viscous component* of the force. The analysis below demonstrates on an example how the difference in the dipole terms of the two equations leads to considerably different predictions of radiated sound amplitude.

At the same time, it is clear that, for an inviscid fluid flow without vorticity, both equations produce the same result for the generated sound pressure.

STATEMENT OF THE PROBLEM

Spatial configuration of the plate and its vibrations

Consider a rigid infinitely thin rectangular plate of sizes, $a \times b$ in y_1- and y_2-directions respectively, vibrating in its own plane along the y_1-axis with the temporal velocity dependence, $U = U_0 e^{-\mathrm{i}\omega t}$, in a viscous fluid with viscosity, μ (Figure 1).

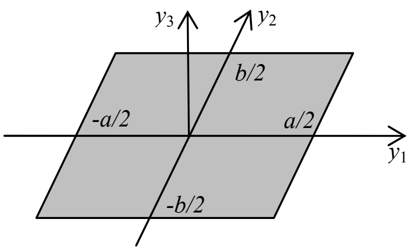

Figure 1.

The goal is to calculate the complex amplitude and spatial configuration of the sound radiated by such vibrations by the two methods.

Existing literature on the subject

The existing literature on the influence of the fluid viscosity on sound radiation by vibrating objects of simple shapes is not abundant. An extensive literature search did not reveal any publications where the problem under consideration (sound radiation by a thin plate) was considered and solved. At the same time, there are a few publications where similar problems were considered.

For example, Ingard and Praidmore-Brown (1955) calculated the sound radiated by a tangentially vibrating plate due to Lighthill's quadrupole sources (first term on the right in Eqs. (8) and (11)). However, this paper is difficult to utilise as it is very short and many intermediate steps and results are missing. The authors calculated that the sound amplitude is proportional to $1/x^2$ and, therefore, is not significant in far field. Also, they did not consider the influence of the plate edges on the radiated sound.

Blinova and Kozhin (1970) calculated the sound radiated by an oscillating cylinder in a viscous medium. They found that, if the cylinder radius is much smaller than the acoustic wavelength, the fluid viscosity leads to an increase in sound radiation corresponding to an increase in cylinder radius by 2δ, where δ is the thickness of the viscous boundary layer.

Assumptions

The following assumptions are made in this paper. These assumptions, while significantly simplifying the necessary mathematical calculations, do not distort the physical nature of the problem.

a) The plate and the region in the fluid containing acoustic sources are considered to be acoustically small:

$$ka << 1, \quad kb << 1, \quad ky << 1. \tag{12}$$

b) The acoustic field is to be found far from the plate:

$$kr >> 1. \tag{13}$$

c) The boundary layer thickness, δ, is much smaller than the size of the plate:

$$\delta << a, \delta << b. \tag{14}$$

d) The amplitude of the plate displacement is much smaller than the boundary layer thickness:

$$U_0 / \omega << \delta. \tag{15}$$

e) The influence on the sound radiation of the side edges of the plate is neglected.

d) The fluid velocity is much smaller than the sound speed, c_0:

$$u << c_0, \tag{16}$$

so that all non-linear terms containing $u_i u_j$ are neglected.

Fluid motion near the plate

Due to Eqs. (14) and (15), the fluid can be split into three distinct regions (Figure 2).

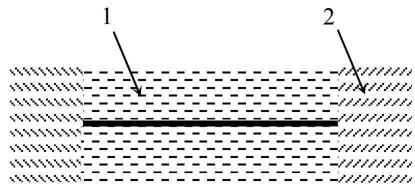

Figure 2. Three regions in the fluid. 1 – the boundary layer; 2 – the transitional region; 3 – the rest of the fluid. The figure is stretched in the vertical direction for better clarity.

First, the fluid motion directly above and below the plate can be assumed to be exactly the same as if the plate were infinite. Note that this assumptions is possible only because there is no fluid flow on average. Equations of the fluid motion for an infinite plate have been solved analytically (Stokes 1851, Landau & Lifshitz 1959). The solution shows the existence of quickly decaying transversal waves propagating normally to the plate. The fluid velocity has only one non-zero component. This component is parallel to the plate and determined as follows:

$$u_1 = U_0 \, \mathrm{e}^{\frac{|y_3|}{\delta}(\mathrm{i}-1)}, \tag{17}$$

where the boundary layer thickness, δ, is

$$\delta = \sqrt{\frac{2\mu}{\rho_0 \omega}}. \tag{18}$$

It can be clearly seen that the viscous wave amplitude decays by the factor of e ≈ 2.718 over δ and by the factor of $\mathrm{e}^{2\pi} \approx 540$ over the spatial period of the wave.

It is known that the viscous fluid flow near oscillating objects becomes potential (free of vortices) at distances of the order of the boundary layer thickness, δ (Landau & Lifshitz 1959, Blinova & Kozhin 1970). Therefore, the fluid outside of the boundary layer can be separated into the fluid volume with potential motion and a narrow transitional region with the width, $\varepsilon \sim \delta$. Contrary to the boundary layer, the fluid velocity in these regions is not known.

In a strict mathematical sense, both the boundary layer and the transitional region are infinite in the vertical direction, as Eq. (17) is determined for $-\infty < y_3 < \infty$. Nevertheless, since the vorticity quickly decreases with increasing $|y_3|$, the vertical thickness of these two regions can still be considered small.

APPLICATION OF KIRCCHOFF EQUATION TO THE PLATE VIBRATIONS

Sound radiation due to the monopole and dipole sources on the plate surface

First consider the sound radiated by the monopole and dipole sources on the surface of the plate. These sources are described by the second and third terms in the right-hand part of Eq. (8) respectively.

Eq. (8) shows that the monopole term is determined by the normal component of the potential fluid velocity, which is proportional to the normal derivative of pressure (Eq. (7)). Landau & Lifshitz (1959) stated that, in the boundary layer near a tangentially vibrating rigid surface, the pressure is constant. Therefore, the monopole term in Eq. (8) vanishes.

The dipole term in Eq. (8) is determined by the pressure on the surface. This pressure represents the normal component of the force acting upon the plate. As the fluid flow is symmetric with respect to the (x_1,x_2) axis, the total normal component of the force is zero, and the dipole term also vanishes. This conclusion can be confirmed by direct calculations of the integral in the third term on the right in (8). These calculations are omitted in this paper.

Sound radiation due to Lighthill's sources in the boundary layer

The sound radiated by Lighthill's quadrupole sources in the boundary layer is described by the first term on the right in Eq. (8). The calculations require the knowledge of Lighthill's stress tensor, T_{ij}. As the plate is considered to be acoustically small (Eq. (12)), the fluid can be considered incompressible in the vicinity of the plate, and, as a result, T_{ij} can be written as follows (Lighthill 1952, Granger 1995):

$$T_{ij} = -\mu\left(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i}\right). \tag{19}$$

In the boundary layer, only one velocity component tangential to the plate is not zero (Landau & Lifshitz 1959):

$$u_1 = U_0 e^{\frac{|y_3|}{\delta}(\mathrm{i}-1)}, \quad u_2 = u_3 = 0. \tag{20}$$

It follows from Eqs (19) and (20) that components of T_{ij} are determined as

$$\begin{aligned} &T_{13} = T_{31} = -U_0\mu\frac{\mathrm{i}-1}{\delta}\mathrm{e}^{\frac{y_3}{\delta}(\mathrm{i}-1)}, y_3 > 0, \\ &T_{11} = T_{22} = T_{33} = T_{23} = T_{32} = T_{12} = T_{21} = 0. \end{aligned} \tag{21}$$

It is clear from Eq. (21) that the double derivative $\partial^2 T_{ij} \big/ \partial y_i \partial y_j$ is equal to zero for any i,j. Therefore, Lighthill's quadrupole sources within the boundary layer do not radiate any sound.

Sound radiation due to Lighthill's sources in the transitional region

The task of calculating the sound radiated by sources in the transitional region is not straightforward, as the spatial velocity distribution in this region is unknown. The exact calculation of this sound may require utilising computational fluid dynamics (CFD) in order to find the unknown velocity. However, the assumptions made in this paper allow completing the calculations without the exact knowledge of the velocity distribution.

It is shown above that no sound is generated on the surface of the plate and in the boundary layer. Therefore, all sound generated by the plate motion is originated in the transitional region. This sound can be found by Kirchhoff integrals over a closed surface which encloses the transitional region. For example, such a surface can be the boundary between the region and the rest of the fluid where the motion is potential (the boundary between the regions 2 and 3 in Figure 2). If this boundary, which is external to the transitional region, is denoted S_{ext}, the sound generated by the plate motion can be determined as follows:

$$\begin{aligned} &p'_K(\mathbf{x}) = \\ &-\frac{1}{4\pi}\frac{\partial}{\partial t}\iint_{S_{ext}} \rho_0 u_n^p(\mathbf{y})\frac{\mathrm{e}^{\mathrm{i}kr}}{r}dS - \frac{1}{4\pi}\iint_{S_{ext}} p'(\mathbf{y})\frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right)dS. \end{aligned} \tag{22}$$

The following important argument can be made about Eq. (22) and the radiated sound which it describes. As the plate is acoustically small (Eq. (12)), the radiated sound does not depend on fine details of spatial distributions of the velocity and pressure on the boundary S_{ext}. Instead, the sound is determined by the total volume velocity and the total force at the boundary. In addition, since the transitional region is thin, it can be assumed that the total volume velocity and the total force on both sides of the region are equal and, therefore, Eq. (22) can be rewritten for the boundary, S_{int}, between the boundary layer and the transitional region (Figure 2):

$$\begin{aligned} &p'_K(\mathbf{x}) = \\ &-\frac{1}{4\pi}\frac{\partial}{\partial t}\iint_{S_{\mathrm{int}}} \rho_0 u_n^p(\mathbf{y})\frac{\mathrm{e}^{\mathrm{i}kr}}{r}dS - \frac{1}{4\pi}\iint_{S_{\mathrm{int}}} p'(\mathbf{y})\frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right)dS. \end{aligned} \tag{23}$$

The integrals in Eq. (23) can be easily calculated, as the distributions of velocity and the pressure on the boundary S_{int} are known. These distributions coincide with the distributions in the boundary layer, since the pressure and the velocity are continuous on S_{int}.

The monopole sources are described by the first integral on the right in Eq. (23). Taking into account Eq. (7), the monopole component of the acoustic pressure can be re-written as

$$p'_{mon}(\mathbf{x}) = \rho_0\frac{1}{4\pi}\mathrm{i}\,\omega\iint_{S_{int}} u_n^p(\mathbf{y})\frac{\mathrm{e}^{\mathrm{i}kr}}{r}dS. \tag{24}$$

The boundary S_{int} consists of two strips at both edges of the plate. The strips lie in the plane (y_2, y_3) and have the width, b, in the y_2-direction. As stated above, even though the strips are infinite in y_3-direction, this does not affect the validity of the assumptions that the boundary layer is thin and the region containing sound sources is acoustically small, as the vorticity decays quickly with increasing y_3 (Eqs. (17), (18)). If these considerations are taken into account, Eq. (24) takes the form of

$$p'_{mon}(\mathbf{x}) = \frac{\mathrm{i}\,\omega\rho}{4\pi}\times \int_{-b/2}^{b/2}\int_{-\infty}^{\infty}\left\{\left[u_n^p(\mathbf{y})\frac{\mathrm{e}^{ikr}}{r}\right]\Bigg|_{y_1=-a/2} + \left[u_n^p(\mathbf{y})\frac{\mathrm{e}^{ikr}}{r}\right]\Bigg|_{y_1=a/2}\right\}dy_2dy_3. \quad (25)$$

The velocity distribution at the surface S_{int} is determined by Eq. (20) for the boundary layer. Although this velocity distribution is not potential, it is the cause of the volume velocity that is responsible for sound radiation. Therefore, the velocity distribution in the boundary layer can be used in Eq. (25) as u_n^p.

It can be shown that Eqs. (12) and (13) lead to the following approximation:

$$\frac{\mathrm{e}^{\mathrm{i}kr}}{r} \approx \frac{\mathrm{e}^{\mathrm{i}kx}}{x}\left(1-\mathrm{i}\,k\sum_{j=1}^{3}y_j\cos\theta_j\right), \quad \cos\theta_j = \frac{x_j}{x}. \quad (26)$$

The normal component of the velocity on both sides of the transitional region needs to be taken with respect to the direction of the normal to the boundary of the region. Therefore, the following equation is satisfied:

$$u_n^p(-a/2, y_2, y_3) = -u_n^p(a/2, y_2, y_3), \quad (27)$$

With the substitution of Eqs. (20), (26) and (27) into Eq. (25) the following equation for p'_{mon} can be obtained:

$$p'_{mon}(\mathbf{x}) \approx -U_0\frac{\omega\rho_0}{4\pi}\frac{\mathrm{e}^{\mathrm{i}kx}}{x}ka\cos\theta_1\int_{-b/2}^{b/2}\int_{-\infty}^{\infty}e^{\frac{|y_3|}{\delta}(\mathrm{i}-1)}dy_2dy_3. \quad (28)$$

After taking the integral and using Eq. (18) for the boundary layer thickness Eq. (28) leads to the final equation for the sound pressure amplitude radiated by the monopole sources:

$$p'_{mon}(\mathbf{x}) = -U_0\frac{1}{4\pi}\rho_0c_0k^2ab\delta\frac{\mathrm{e}^{\mathrm{i}kx}}{x}(\mathrm{i}+1)\cos\theta_1. \quad (29)$$

Note that the radiation has dipole directivity with the dipole axis coinciding with the direction of the plate vibrations.

The second term in the right-hand part of Eq. (23) represents the dipole sources due to the pressure at the boundary S_{int}. The pressure acting upon the transitional region from the boundary layer is determined by the corresponding component of the stress tensor, p_{ij} (Eq. (3)):

$$p'(\mathbf{y}) = p_{13}(\mathbf{y}) = -\mu\frac{\partial u_1(\mathbf{y})}{\partial y_3}. \quad (30)$$

The substitution of Eq. (17) for u_1 to Eq. (30) leads to

$$p'(\mathbf{y}) = -\mu U_0\frac{\mathrm{i}-1}{\delta}\mathrm{e}^{\frac{y_3}{\delta}(i-1)}, \; y_3 > 0. \quad (31)$$

It can be shown that, in far field of an acoustically small object (Eqs. (12) and (13)), the following approximation is satisfied:

$$\frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right) = \mp\frac{\partial}{\partial y_1}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right) \approx \pm\mathrm{i}\,k\frac{\mathrm{e}^{\mathrm{i}kx}}{x}\left[\cos\theta_1\left(1-\mathrm{i}\,k\sum_{j=1}^{3}y_j\cos\theta_j\right)-\frac{y_1}{x}\right], \quad (32)$$

where the top and bottom signs correspond to $y_1 = \pm a/2$.

The substitution of Eqs. (31) and (32) into the second term in Eq. (23) results in the following equation for the sound amplitude generated by the dipole sources:

$$p'_{dip}(\mathbf{x}) = -\frac{1}{4\pi}U_0\mu\frac{\mathrm{i}-1}{\delta}\mathrm{i}\,k\frac{\mathrm{e}^{\mathrm{i}kx}}{x}\times \int_{-b/2}^{b/2}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{|y_3|}{\delta}(\mathrm{i}-1)}\left\{\left[\cos\theta_1\left(1-\mathrm{i}\,k\sum_{j=1}^{3}y_j\cos\theta_j\right)-\frac{y_1}{x}\right]\Bigg|_{y_1=a/2} + \left[\cos\theta_1\left(1-\mathrm{i}\,k\sum_{j=1}^{3}y_j\cos\theta_j\right)-\frac{y_1}{x}\right]\Bigg|_{y_1=-a/2}\right\}dy_2dy_3. \quad (33)$$

Taking the integrals in Eq. (33) leads to the following equation for the pressure amplitude of the sound wave generated by dipole sources:

$$p'_{dip}(\mathbf{x}) = \frac{1}{2\pi}U_0\,\mathrm{i}\,\rho_0c_0k^2b\delta^2\frac{\mathrm{e}^{\mathrm{i}kx}}{x}\cos\theta_1. \quad (34)$$

Eqs. (29) and (34) clearly show that the sound due to forces, $p'_{dip}(\mathbf{x})$, and the sound due to the volume velocity, $p'_{mon}(\mathbf{x})$, have the directivity of a dipole. The ratio of the amplitudes of these two components of the sound is determined by

$$\left|\frac{p'_{dip}}{p'_{mon}}\right| = \frac{\delta\sqrt{2}}{a}. \quad (35)$$

It is clearly seen that, due to Eq. (14), $|p'_{dip}| << |p'_{mon}|$.

Eqs (23), (29) and (34) lead to the following equation for the acoustic pressure calculated by non-uniform Kirchhoff's equation:

$$p'_K(\mathbf{x}) = p'_{mon}(\mathbf{x}) + p'_{dip}(\mathbf{x}) = \frac{1}{2\pi}U_0\rho_0c_0k^2b\delta\frac{\mathrm{e}^{\mathrm{i}kx}}{x}\left(-\frac{1}{2}a(\mathrm{i}+1)+\mathrm{i}\,\delta\right)\cos\theta_1. \quad (36)$$

APPLICATION OF FW-H EQUATION TO THE PLATE VIBRATIONS

In the FW-H equation (Eq. (11)), the first term on the right describes the quadrupole sources and is identical to the corresponding term in the non-uniform Kirchhoff equation. It is shown above that the sound amplitude determined by this term is, in fact, the total sound amplitude predicted by the non-uniform Kirchhoff equation (Eq. (36)).

The monopole sources in the FW-H equation are described by the second term on the right. It is clear that this term vanishes, as the normal component of the velocity of the plate is zero (the plate vibrates in its own plane). Therefore, the sound radiated by the monopole sources also vanishes.

The total force, **P**, in the third (dipole) term of the FW-H equation, includes both the normal component (pressure) and the viscous component of the force. It is clear from Eq. (11) that the normal component of the force does not generate any sound, as the forces acting upon the two sides of the plate cancel each other.

Therefore, the amplitude of the sound predicted by the FW-H equation is determined as follows:

$$p'_{visc}(\mathbf{x}) = -\frac{1}{4\pi}\iint_S P_1(\mathbf{y})\frac{\partial}{\partial y_1}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right)dS, \tag{37}$$

and y_1-component of the force, $P_1(\mathbf{y})$, can be expressed through the stress tensor (Landau & Lifshitz 1959):

$$P_1(\mathbf{y}) = p_{13}(\mathbf{y}) = -\mu\frac{\partial u_1}{\partial u_3} = -\mu U_0\frac{\mathrm{i}-1}{\delta}\mathrm{e}^{\frac{|y_3|}{\delta}(i-1)}. \tag{38}$$

Inserting Eqs. (38) and (32) into Eq. (37) one can obtain the following equation for the sound radiated due to viscous forces according to the FW-H equation:

$$p_{visc}(\mathbf{x}) = \frac{1}{4\pi}U_0\rho_0 c_0 k^2 ab\delta(\mathrm{i}+1)\frac{\mathrm{e}^{\mathrm{i}kx}}{x}\cos\theta_1. \tag{39}$$

As stated above, the terms describing the sound radiated by Lighthill's quadrupole sources in the fluid volume are identical in both equations and determined by Eq. (36). Therefore, adding together Eq. (36) and Eq. (39) leads to the following equation for the sound pressure amplitude as predicted on the basis of the FW-H equation:

$$p'_{FW-H}(\mathbf{x}) = \frac{\mathrm{i}}{2\pi}U_0\rho_0 c_0 k^2 b\delta^2\frac{\mathrm{e}^{\mathrm{i}kx}}{x}\cos\theta_1. \tag{40}$$

Note that the two added terms partially cancel each other.

COMPARISON OF THE TWO EQUATIONS FOR THE RADIATED SOUND

Eqs. (36) and (40) represent the predictions for the radiated sound amplitude obtained on the basis of non-uniform Kirchhoff and FW-H equations respectively. It can be clearly seen that, due to the assumption that the boundary layer is thin (Eq. (14)), the prediction of the FW-H equation is much smaller in amplitude than that of the non-uniform Kirchhoff equation. Phases of both sound waves are also different.

Whereas the current analysis does not provide direct confirmation or otherwise for any of these two formulations, it is the opinion of the present author that non-uniform Kirchhoff equation should take preference due to the following two factors.

First, the non-uniform Kirchhoff equation is simply the most general solution of Lighthill's wave equation in the presence of solid boundaries, and Lighthill's equation is derived on the basis of the most general mass and momentum conservation laws of fluid motion. The FW-H equation is also derived on the basis of the conservation laws, but these laws are first considered at a boundary with discontinuity in velocity and stresses and then applied to a rigid boundary. It is the view of the present author that this application of the conservation laws is questionable, but a re-consideration of this issue is outside the scope of this article.

The second factor in favour of the non-uniform Kirchhoff equation is a more physically justified mechanism of sound generation. As shown above, according to the non-uniform Kirchhoff formulation, the sound is generated by Lighthill's quadrupole sources in the transitional region between the boundary layer and the rest of the fluid. This mechanism of sound generation appears to be more physically rational than the one assumed in the FW-H formulation, where the sound is generated directly by tangential viscous stresses on the plate. Whereas undoubtedly these stresses are the ultimate source of the acoustic energy transferred from the vibrating plate to the fluid, they cannot compress the surrounding fluid, and, therefore, can radiate sound only indirectly through the interaction between the viscous boundary layer generated by them and the rest of the fluid.

POSSIBLE EXPERIMENTAL VERIFICATION OF THE OBTAINED RESULTS

As both formulations predict significantly different sound amplitudes for the case under consideration, conducting direct measurements of the sound radiated by such a vibrating plate could prove useful for verification of these methods. However, such measurements may be difficult due to very low intensity of the radiated sound. Therefore, other experimental arrangements may be necessary for this purpose.

The results obtained in this paper demonstrate that, apart from being significantly different in the absolute value, the acoustic wave amplitudes predicted by the FW-H and non-uniform Kirchhoff formulations are different in phase (Eqs. (36) and (40)) due to their different physical origin. Therefore, it may be helpful for the purpose of verification to compare not the amplitudes or intensities, but the phases of the predicted and measured sound wave relative to the plate motion.

The anechoic wind tunnel experiment with a vortex street by Leclercq and Doolan (2009) can be considered analogous to the plate vibrations described here. In both cases, the sound is generated due to the interaction between a rotational fluid flow, a solid object, and the rest of the fluid. Therefore, due to this analogy, it can be expected that the phases of the sound waves predicted by the two equations will also be different. As a result, measurements of the phase of the acoustic signal may provide a way to prove or disprove the FW-H or other formulations for prediction of the sound radiated by a fluid flow.

RECOMMENDATIONS FOR APPLICATION OF THE FW-H AND NON-UNIFORM KIRCHHOFF EQUATIONS

On the one hand, there are situations where the FW-H and non-uniform Kirchhoff equations produce identical results. For example, a comparative study of these two formulations has been published by Brentner and Farassat (1997), who concluded that both formulations were equivalent in linear and inviscid flow. The analysis above shows that, indeed, these two conditions lead to the equivalence of the third (dipole) term in the FW-H equation (Eq. (11)) and the non-uniform Kirchhoff equation (Eq. (8)).

At the same time, for the second (monopole) terms in the two equations to be equivalent, one more condition should be satisfied. Namely, the fluid flow should be purely potential, i.e. it should not contain any vorticity. This is clear from Eq. (8), where the monopole term is not simply the *total* velocity of the fluid, but the velocity that is determined through the pressure gradient (Eq. (7)), i.e. the potential velocity.

It can be concluded that both equations are equivalent in a linear and potential flow. Consequently, both of them are applicable to tasks of linear acoustics, including sound scattering and generation problems which do not involve turbulent (rotational) flow.

At the same time, in most real-world situations of sound generation by a fluid flow, the flow is turbulent and, therefore, vorticity cannot be neglected. In such situations, the non-uniform Kirchhoff equation can be recommended for use as a better established and justified formulation.

The influence of flow non-linearity on the sound radiation is not considered in this analysis, as it is carried out for a linear flow (Eq. (16)).

As for practical recommendations how to control the sound produced by the mechanism considered in this paper, it can be concluded from Eqs. (18) and (36) that the sound amplitude is proportional to the area of the plate, to the wavenumber (i.e. frequency) in the power 3/2, and to the square root of the fluid viscosity. Therefore, changing these parameters will correspondingly affect the radiated sound amplitude.

CONCLUSIONS

In this article, the Ffowcs Williams and Hawkings (FW-H) and non-uniform Kirchhoff formulations for the evaluation of aerodynamic sound are considered. It is shown that the two formulations contradict each other. This contradiction is due to the fact that the latter formulation takes into account only normal pressure on the surface, whereas the former formulation also includes viscous tangential forces on the surface.

Both formulations are applied to sound radiation by a thin rigid plate vibrating in its own plane in a viscous fluid. A few simplifying assumptions are utilised in the analysis. Calculation by means of the non-uniform Kirchhoff equation show that all sound sources are concentrated in a thin transitional region between the viscous boundary layer and the rest of the fluid. In addition to these sources, which are identical in both formulations, in the FW-H formulation viscous forces on the boundary also produce sound. When the two components of the sound are added together, they partially cancel each other and, as a result, the sound amplitude predicted by the FW-H equation is much smaller than that predicted by the non-uniform Kirchhoff equation.

As the sound waves predicted by these two equations are different in phase, it is suggested that the experimental verification of both formulations can be done by measuring the phase of the sound instead of its amplitude. Such measurements can be carried out using an existing experimental setup.

It is recommended that, due to its better physical justification, the non-uniform Kirchhoff formulation should be used for calculation of aerodynamic sound generated by a turbulent flow. In linear potential flow both formulations are equivalent.

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