Acoustic-Gravity Waves, Theory and Applications

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Abstract

Acoustic-gravity waves (AGWs) are critical to understanding a wide range of oceanographic and geophysical phenomena, including tsunamis and microseisms. These waves, governed by the interplay between gravity, compressibility, and elasticity, propagate through the ocean and atmosphere, transporting energy over long distances with minimal attenuation. Recent advancements in AGW theory, including resonant triad interactions and the effects of seabed elasticity, have provided deeper insights into their behaviour and practical applications. This paper presents the theoretical foundations of AGWs and highlights their role in early warning systems, tsunami mitigation, and potential energy harnessing.

1 Introduction

Acoustic-gravity wave (AGW) theory concerns the propagation of wave disturbances in a compressible medium, such as the ocean or the atmosphere, under the effects of gravity. These waves, which can traverse vast distances in both the ocean and atmosphere, are key to understanding a wide range of oceanographic and geophysical processes, including tsunamis and microseisms. The study of nonlinear AGWs has expanded significantly since their initial exploration by Longuet-Higgins (1950), who examined the role of gravity in surface wave interactions that generate microseisms. More recent research has incorporated the effects of compressibility, elasticity, and higher nonlinear dynamics to develop a deeper understanding of AGW behaviour (Kadri and Akylas 2016, Kadri and Stiassnie 2013, Williams and Kadri 2023).

The propagation of AGWs is governed by both the gravitational forces acting on the ocean surface and the compressibility of the water, which allows acoustic waves to propagate through the medium. The combination of these forces results in wave modes that are distinct from traditional surface gravity waves, with AGWs being capable of oscillating vertically throughout the ocean's depth. This oscillatory behaviour introduces vertical stresses on the seabed, leading to energy transfer between the ocean and the solid Earth, a phenomenon that contributes to the generation of microseisms.

In addition to their geophysical significance, AGWs offer practical applications in areas such as tsunami early warning systems and environmental monitoring. Tsunamis, triggered by underwater earthquakes or volcanic eruptions, can generate AGWs that propagate much faster than the tsunami itself. These waves have the potential to provide an early warning signal for coastal regions,

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allowing for more timely evacuations and disaster preparedness (Kadri and Stiassnie 2013, Mei and Kadri 2018, Nosov 1999, Yamamoto 1982). The ability of AGWs to travel long distances with minimal energy loss makes them ideal candidates for monitoring oceanic disturbances in real time. In addition, they can be used for locating objects, such as meteorites or airplanes, impacting at the sea surface (Kadri 2019, 2024, Kadri et al. 2017).

Furthermore, the nonlinear interaction of AGWs with other wave modes, particularly through resonant triad interactions, has been a topic of growing interest. Kadri and Akylas (2016) demonstrated that AGWs could engage in triad resonances with surface gravity waves, resulting in significant energy transfer between these modes. In theory, this resonant mechanism plays a crucial role in energy redistribution during events such as tsunamis and could potentially be harnessed for energy generation (Kadri 2017). It also explains the evolution of Faraday-type waves, that may have implication in pilot-wave quantum analogue systems Kadri and Wang (2021).

Recent advancements in AGW research have also highlighted the importance of seabed elasticity in wave propagation. The presence of an elastic seabed modifies the wave modes that can propagate within the system, revealing the connection to additional wave types, such as *Scholte* and *Rayleigh* waves, which propagate along the ocean floor and on land (Eyov et al. 2013, Williams and Kadri 2023). This interaction between the ocean and the elastic seabed introduces complexities in the dispersion relations that govern wave behaviour, particularly in shallow water environments, where the effects of compressibility and elasticity are most pronounced on tsunami phase velocity, whereas gravitational effects in the solid earth are of a secondary importance (Abdolali et al. 2019).

This brief review aims at providing an overview of the theory and a selection of applications of AGWs, exploring their role in oceanographic processes and their potential for practical implications. The theoretical foundations of AGW propagation, including the effects of compressibility, elasticity, and nonlinear dynamics are discussed, and their implications for tsunami mitigation, energy harnessing, and environmental monitoring are examined. Through this synthesis, the hope is to contribute to the ongoing development of AGW-based technologies and deepen our understanding of these critical wave phenomena.

2 Preliminaries

Consider the propagation of wave disturbances under the combined action of gravity and compressibility in water of constant depth h, and a rigid sea bottom ($z = -h$). The water is assumed to be an inviscid barotropic fluid (where the pressure p is a function of the density ρ only) with constant sound speed $c \equiv (\mathrm{d}p/\mathrm{d}\rho)^{1/2}$. Furthermore, the motion is assumed to be irrotational. Following Kadri and Akylas (2016) we define a non-dimensional parameter $\mu = gh/c^2$, where g is the gravitational acceleration. The parameter μ controls the effects of gravity relative to compressibility, and for the range of water depths concerned $\mu \ll 1$. Thus, the acoustic phase speed far exceeds the free surface-gravity wave phase speed, and both wave types may feature disparate spatial and/or temporal characteristic scales. In addition, we consider μh as lengthscale and h/c as timescale.

The assumption of irrotationality allows us to formulate the problem in terms of the velocity potential $\varphi(x, z, t)$, where $\mathbf{u} = \nabla\varphi$ is the velocity field. The field equation governing φ in the fluid interior is obtained by combining continuity with the unsteady Bernoulli equation as in Longuet-

Higgins (1950), which gives

$$\frac{1}{\mu^2}(\varphi_{xx}+\varphi_{zz})-\varphi_{tt}-\varphi_z-|\nabla\varphi|_t^2-\tfrac{1}{2}\mathbf{u}\cdot\nabla\left(|\nabla\varphi|^2\right)=0. \tag{1}$$

Equation (1) is the nonlinear wave equation. To complete the boundary-value problem, a free-surface boundary assumption is made so that the usual kinematic and dynamic conditions apply on $z=\eta(x,t)$. Expanding the two free-surface conditions about $z=0$, expressing η in terms of φ to cubic order of approximation, and eliminating it yields

$$\begin{aligned}\varphi_{tt}+\varphi_z+|\nabla\varphi|_t^2-\left\{\varphi_t(\varphi_{tt}+\varphi_z)\right\}_z+\tfrac{1}{2}\mathbf{u}\cdot\nabla\left(|\nabla\varphi|^2\right)-\left\{\varphi_t|\nabla\varphi|_t^2\right\}_z&\\ -\tfrac{1}{2}\left\{(\varphi_{tt}+\varphi_z)\left(|\nabla\varphi|^2-\varphi_t^2\right)\right\}_z=0 \qquad (z=0).&\end{aligned} \tag{2a}$$

Finally, the no-penetration boundary condition on the rigid bottom at $z=-1/\mu$ is

$$\varphi_z=0 \qquad (z=-1/\mu). \tag{2b}$$

In the limit $\mu \ll 1$, equation (1) becomes the Laplace equation and the boundary conditions (2) reduce to the classical incompressible deep-water wave problem (correct to cubic terms). For the gravity waves the chosen lengthscale μh pertains to deep-water with free surface, where the gravity pressure field decays exponentially with depth. On the other hand, the acoustic waves oscillate within the entirety of the fluid depth, and thus x and z have to be re-scaled accordingly.

2.1 Gravity wave dispersion relation

Dropping the nonlinear terms in (1) and (2a), we are concerned with wave modes that propagate along the x-axis with wavenumber k and frequency ω. We apply a separation of variables in the form (following Kadri and Akylas (2016))

$$\varphi=f(z)\exp\left(\tfrac{1}{2}\mu^2 z\right)\exp\left\{\mathrm{i}(kx-\omega t)\right\}. \tag{3}$$

Substituting (3) into (1)–(2), and taking $k=O(1)$, the solution decays exponentially with depth,

$$f=\mathrm{e}^{|k|z}+O(\mu^2), \tag{4}$$

and thus the expected dispersion relation for the gravity wave mode in deep water is recovered,

$$\omega^2=|k|+O(\mu^4). \tag{5}$$

2.2 Acoustic wave dispersion relation

For the acoustic mode the vertical profile $f(z)$ becomes oscillatory in the low-wavenumber limit, $k^2<\mu^2\omega^2$, which requires the re-scaling $k=\mu\kappa$, $Z=\mu z$. Here, the water depth is O(1), and assuming $\Omega^2=\omega^2-\kappa^2>0$, it follows that

$$f=\cos\Omega(Z+1)-\frac{\mu}{2\Omega}\sin\Omega(Z+1)+O(\mu^2), \tag{6}$$

where

$$\cos\Omega + \mu\frac{\Omega^2-\kappa^2}{2\Omega\omega^2}\sin\Omega = O(\mu^2). \tag{7}$$

Equation (7) shows that, unlike the gravity dispersion relation where only a single propagating mode exists, here there is a countable infinity of propagation modes that obey the dispersion relations

$$\omega^2 = \omega_n^2 + \kappa^2 + \mu\frac{\omega_n^2-\kappa^2}{\omega_n^2+\kappa^2} + O(\mu^2) \qquad (n = 0, 1, 2, \ldots), \tag{8}$$

where $\omega_n = \left(n+\frac{1}{2}\right)\pi$. Each acoustic mode can propagate along x ($\kappa^2 > 0$) only if $\omega > \omega_n^c$, where

$$\omega_n^c = \omega_n + \frac{\mu}{2\omega_n} + O(\mu^2) \tag{9}$$

is the corresponding cut-off frequency. Note that, to leading order the dispersion relations (8) agree with those of pure acoustic waves in a fluid layer bounded by a rigid bottom and a free surface.

2.3 Elastic effects

The elasticity of the seabed plays a significant role in the behaviour of AGWs, influencing both their propagation characteristics and interaction with the ocean floor. When considering the propagation of AGWs over an elastic seabed, the ocean's compressibility and the elasticity of the solid layer beneath introduce complexities that modify the wave dynamics compared to models with rigid seabeds. In the framework of elasticity, the interaction between the ocean and the elastic seabed is governed by both the pressure and shear waves within the seabed. These interactions are described by the potentials of the velocity field in the liquid (φ_l) and solid (φ_s, ψ_s) as given in the main equations

$$\begin{aligned}
\nabla^2\varphi_l - \mu_l(\varphi_{l,tt} + \varphi_{l,z}) &= 0, \quad -1 \le z \le 0,\\
\nabla^2\varphi_s - \mu_s(\varphi_{s,tt} + \varphi_{s,z}) &= 0, \quad z \le -1,\\
\nabla^2\psi_s - \mu_s(\psi_{s,tt} + \psi_{s,z}) &= 0, \quad z \le -1,
\end{aligned} \tag{10}$$

where $\mu_i = gh/c_i^2$ (with $i = l, p, s$) are small dimensionless parameters representing the squares of the ratios of surface gravity wave speed to sound, pressure, and shear wave speeds, respectively. The equations are normalised using the water depth h as the length scale, $\sqrt{h/g}$ as the time scale, and densities are normalised by the water density ρ_l. The effect of elasticity on the propagation of AGWs becomes evident in the dispersion relation which can be expressed as

$$r\tanh(r) = \frac{\omega^2(\varepsilon_1+\varepsilon_2)}{\varepsilon_1+\varepsilon_2\omega^4/r^2+\beta\mu_l/2r^2}, \tag{11}$$

where $r^2 = k^2 - \mu_l\omega^2 + \mu_l^2/4$ represents the vertical eigenvalue of the potential φ_l in the water, and k is the horizontal wavenumber. The parameters ε_1, ε_2, and β account for the effects of elasticity and gravity and can be found in Abdolali et al. (2019). It is easy to see that ignoring elasticity and gravity effects the standard dispersion relation is retrieved.

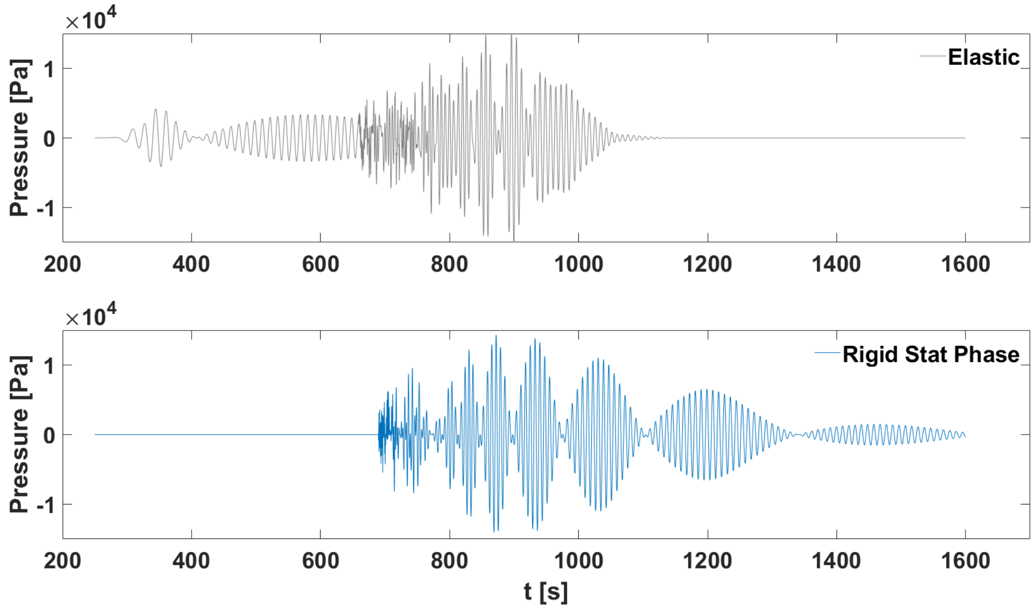

Fig. 1: Top plot, first acoustic mode with elastic seabed, bottom plot with rigid seabed. Credit: reprinted from Figure 11 of Williams and Kadri (2023)

The elasticity of the seabed alters the wave modes that propagate within the system. Unlike the rigid seabed scenario, where gravity waves decay exponentially with depth, the presence of elasticity allows for the generation of AGWs that oscillate vertically down to the seabed. This oscillation introduces vertical stresses that lead to energy transmission between the ocean and the solid seabed, generating phenomena such as microseisms.

Equation (11) encapsulates the dependence of the AGW phase speed (and the tsunami) on parameters such as seabed elasticity, compressibility, gravity in the water and solid layers, and the ocean depth. The inclusion of elasticity leads to additional wave modes, including the *Scholte* wave, which can propagate along the ocean floor, showing that the elasticity significantly influences both shallow and deep water behaviours as illustrated in Figure 1.

3 Generation mechanisms

AGWs can be generated through various mechanisms, including the nonlinear interaction of surface gravity waves (Kadri and Akylas 2016, Kadri and Stiassnie 2013, Longuet-Higgins 1950), submarine earthquakes (Mei and Kadri 2018, Stiassnie 2010, Williams and Kadri 2023, Yamamoto 1982), submerged moving vortices (Dong et al. 2024a,b), surface disturbance pressure (Meza-Valle et al. 2023, Omira et al., Renzi and Dias 2014), airplane crashes (Kadri 2024, Kadri et al. 2017), and underwater explosions. The primary focus of recent studies has been on understanding these generation processes and their dependence on oceanographic conditions. In the sequel, attention is focused on submarine earthquakes and triad resonance.

3.1 Nonlinear interaction of AGWs

Longuet-Higgins (1950) demonstrated, in a landmark paper, that quadratic interactions of gravity surface waves can resonantly excite compression modes in water of finite depth which can explain the generation of microseisms in the ocean. However, the specific configuration considered by Longuet-Higgins (1950) does not take into account propagating acoustic modes. More recently, Kadri and Stiassnie (2013) showed numerically that triad resonance between gravity and acoustic modes is possible when two opposing gravity waves of nearly identical frequencies interact with an acoustic mode of almost double the frequency. This study, combined with further analysis by Kadri and Akylas (2016), explores how these interactions can result in resonant triad formations that transfer energy from gravity waves to AGWs. Particularly, Kadri and Akylas (2016) considered two gravity waves (k_+, ω_+) and (k_-, ω_-) observing the dispersion relation (5), along with an acoustic wave $(\mu\kappa, \omega)$ which satisfies the dispersion relations (8) for some $n = 0, 1, 2, \ldots$. To form a resonant triad, these modes must also obey the resonance conditions

$$\text{(i)} \quad k_+ + k_- = \mu\kappa; \qquad \text{(ii)} \quad \omega_+ + \omega_- = \omega. \tag{12}$$

Then applying multiple-scale analysis they derived the amplitude evolution equations taking into account nonlinear cubic terms. For a slow timescale $T = \mu t$, the amplitude evolution equations for the acoustic mode $A(T)$ and the two gravity waves $S_\pm(T)$ are given by Kadri and Akylas (2016)

$$\frac{\partial A}{\partial T} + \frac{\kappa}{\omega}\frac{\partial A}{\partial X} = \mathrm{i}\gamma A + \delta S_+ S_-; \quad \frac{\partial S_\pm}{\partial T} = -\frac{\delta}{2} A S_\mp^* - \mathrm{i}\lambda\left(S_\pm^2 S_\pm^* - 2|S_\mp|^2 S_\pm\right),^{1} \tag{13}$$

where

$$\gamma = \frac{\kappa^2 - \omega_n^2}{2\omega^3} - \beta; \qquad \delta = \frac{(-1)^n}{4}\omega_n \omega^2 \alpha; \qquad \lambda = \frac{1}{64}\omega^7 \alpha^2 \tag{14}$$

and $*$ stands for complex conjugate. The terms on the right-hand side of (13) account for the quadratic and cubic interactions, which enter the evolution equations at the same order for $\alpha = O(1)$.

The work by Kadri and Akylas (2016) demonstrated that such resonant interactions can result in significant energy transfer from surface waves to acoustic modes, especially in harmonic finely tuned triads where nearly all gravity wave energy converts to AGWs. However, it was found that for wavepackets the energy transfer becomes far less efficient as presented in Figure 2.

3.2 Submarine earthquakes

The significance of slight ocean compressibility in fluid mechanics has been explored extensively, particularly in the context of tsunami generation and the behaviour of AGWs. Early studies by Miyoshi (1954), Sells (1965), Kajiura (1970), and Yamamoto (1982) laid the foundation by examining how compressibility affects tsunami dynamics. Yamamoto (1982) provided an analytical solution for the generation of gravity and AGWs caused by the vertical oscillation of a block on the ocean floor. Later, Nosov (1999), Nosov and Kolesov (2007), Chierici et al. (2010), and Stiassnie (2010) further investigated the effects of slight water compressibility. Stiassnie (2010), in particular, addressed the problem of a sudden rise of a block of the ocean floor, releasing AGWs at various frequencies, and developed an analytical tool to calculate the leading AGW modes. These studies

[1] There is a factor half correction involving the last term of the *rhs* of the equation following Kadri and Wang (2021)

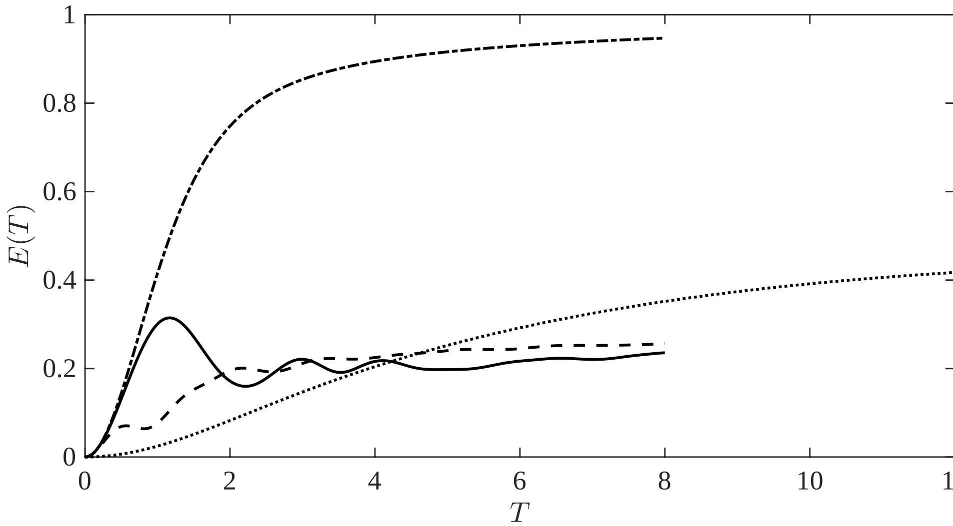

Fig. 2: Time histories of acoustic waves energy, $E(T)$, for certain values of the wave steepness parameter α and the resonance tuning parameter γ. (—): $\alpha = 1$, $\gamma = 4.1$; (– –): $\alpha = 1$, $\gamma = 0$; (· · · ·): $\alpha = 0.2$, $\gamma = 0.14$; (– · –): $\alpha = 1$, $\gamma = 0$ and the cubic terms in equation (13) are ignored (perfectly tuned standard triad). Credit: reprinted from Figure 2 of Kadri and Akylas (2016).

highlighted the potential of using the acoustic forerunner as an early warning signal for tsunamis (Abdolali et al. 2015, Kadri 2015, 2016, Kadri and Akylas 2016, Kadri and Stiassnie 2013, Nosov 1999, Renzi and Dias 2014, Saito et al. 2011, Yamamoto 1982).

Further advancements in understanding AGWs came with the development of finite fault models, which provide a three-dimensional theory of AGWs using the Green's function method (Hendin and Stiassnie 2013). However, the practical utility of these models for real-time predictions is limited due to their computational complexity; the integral form solutions require dividing any considered fault shape into numerous small elements, calculating each element's contribution, and then summing these to obtain the total effect. This method becomes computationally expensive, particularly when modeling complex, multi-fault ruptures commonly observed in nature (Hamling and et al. 2017).

To address these limitations, Mei and Kadri (2018) proposed an alternative approach using multiple scales analysis for slender faults of width $2b$ and length $2L$. Assuming the fault is moving vertically with a constant speed W_0 for a time duration T, and introducing multiple scale coordinates, $x, z, X = \epsilon^2 x, Y = \epsilon y$, where $\epsilon = b/L$, they derived a relation for the sea bottom pressure in the form

$$p = \rho W_0 |A| \frac{2^{7/2} c}{\sqrt{\pi^3 x_0 k}} \sin(kb) \sin\left(\hat{\Omega} T\right) \tag{15}$$

where ρ is the water density; A is the two dimensional envelope (see Mei and Kadri (2018) for details), k is the wave number and $\hat{\Omega}$ is the frequency calculated directly from the pressure signal.

The fault location (x_0, y_0) and eruption time (t_0) are given analytically by

$$x_0 = \frac{(\hat{t}_1 - \hat{t}_2)c}{\left\{1 - \left[\frac{\pi c}{2h\hat{\Omega}_2}\right]^2\right\}^{-1/2} - \left\{1 - \left[\frac{\pi c}{2h\hat{\Omega}_1}\right]^2\right\}^{-1/2}};$$
$$t_0 = \hat{t}_j - \frac{x_0}{c}\left\{1 - \left[\frac{\pi c}{2h\hat{\Omega}_j}\right]\right\}; \qquad y_0 = \sqrt{(t_0 c)^2 - x_0^2} \tag{16}$$

where $\hat{t}_j$ is the measured time for the j-th pressure point in the signal. Though it requires a direct numerical solution to solve for the remaining fault dynamics and geometry – Williams et al. (2021) provided a similar solution for the multi-fault problem (see Figure 3). Additionally, they developed an inverse approach to derive fault location and rupture parameters, which has been further refined in subsequent work by Gomez and Kadri (2021b) and Kadri et al. (2024). These developments provide practical tools for tsunami early warning systems.

4 Applications in the Ocean and beyond - an open discussion

4.1 Early tsunami warning

The application of AGWs as precursors for tsunami was emphasised by Yamamoto (1982) and many subsequent works as mentioned earlier. However, it was the explicit solution of the pressure field derived by Mei and Kadri (2018) that made the real-time assessment of tsunamis practically possible. A recent work by Kadri et al. (2024) presents a methodology for a rapid tsunami warning system that integrates input data from various measurement sources and existing analysis techniques. The system enables real-time mapping of risk areas and potential travel paths once the earthquake epicenter is identified. Machine learning is then used to analyse live acoustic signals to classify earthquake magnitude and strike mode (Gomez and Kadri 2021a). For earthquakes with a vertical strike component, an inverse problem approach (Gomez and Kadri 2023, Kadri et al. 2017, Mei and Kadri 2018) is applied to estimate the probability density function for the fault's geometry and dynamics. These estimates are subsequently fed into an analytical (direct) model (Mei and Kadri 2018, Williams et al. 2021) to calculate tsunami amplitudes in the identified risk areas, and the associated pressure field. The processing time for analysing acoustic data ranges from seconds to a few minutes on a standard multi-core PC, depending on the size of the processed signal. The methodology has been validated with historical earthquake-induced tsunamis (Gomez and Kadri 2023, Kadri et al. 2024). Figure 4 presents screenshots from an operational software called GREAT that employs the above mentioned AGW technology for early tsunami assessment.

4.2 Energy transfer via nonlinear interaction of AGWs

The study of energy transfer through the nonlinear interaction of AGWs has been explored in various contexts, demonstrating how different wave modes interact and exchange energy. In compressible ocean environments, AGWs can be generated through resonant triad interactions, where the energy transfer occurs between gravity waves and acoustic modes. Early foundational work by Longuet-Higgins (1950) established the basis for understanding these nonlinear resonances in the ocean, with

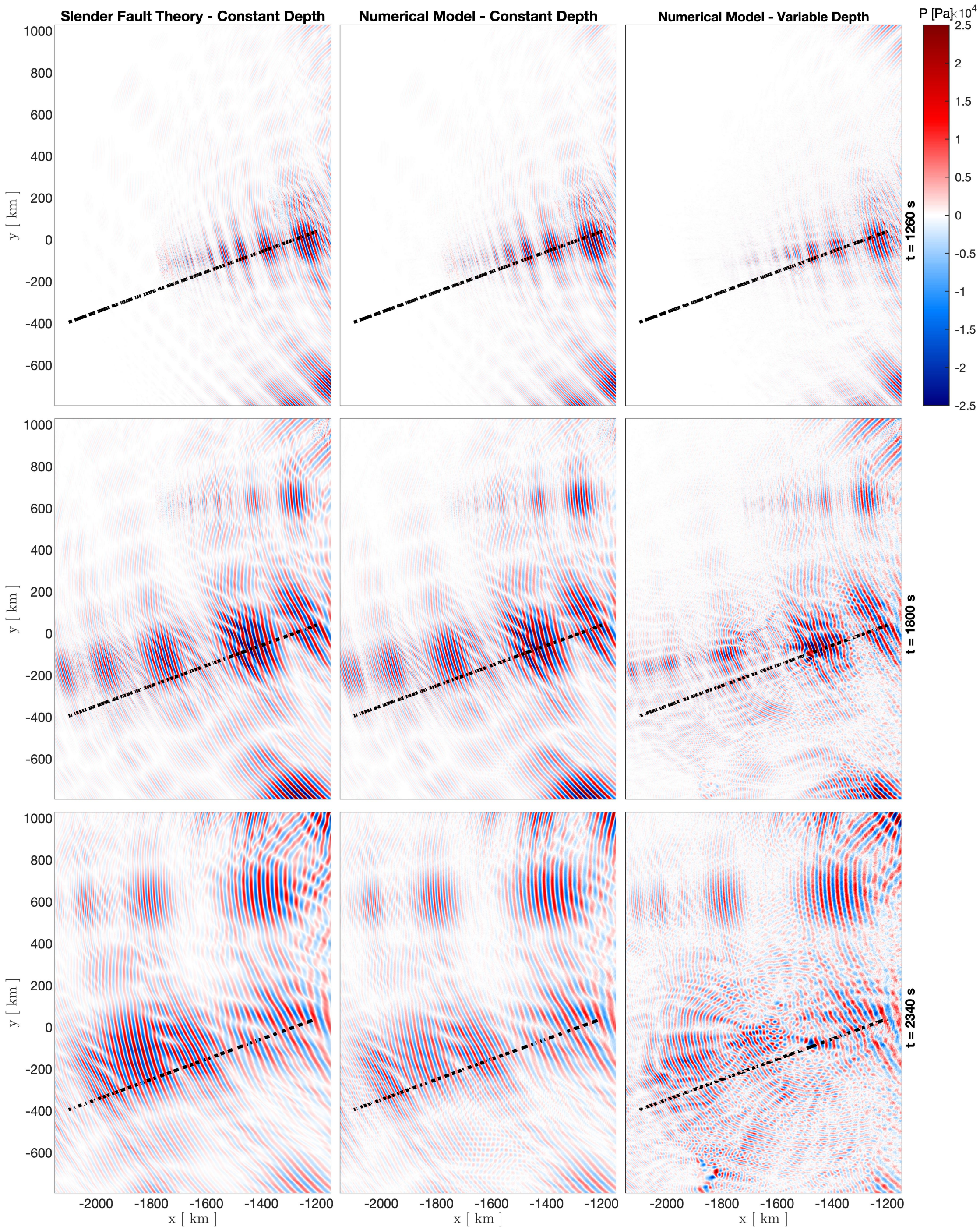

Fig. 3: Snapshots of bottom pressure fields at t=1260 s (top row), t=1800 s (middle row) and t=2340 s (bottom row) from slender fault theory (left column), numerical model for the case of constant depth of 4 km (middle column) and numerical model for the case of variable depth (right column). The dynamic pressure variation is indicated with reference to the colour bar where white corresponds to 0 Pa. Credit: reprinted from Figure 8 of Williams et al. (2021).

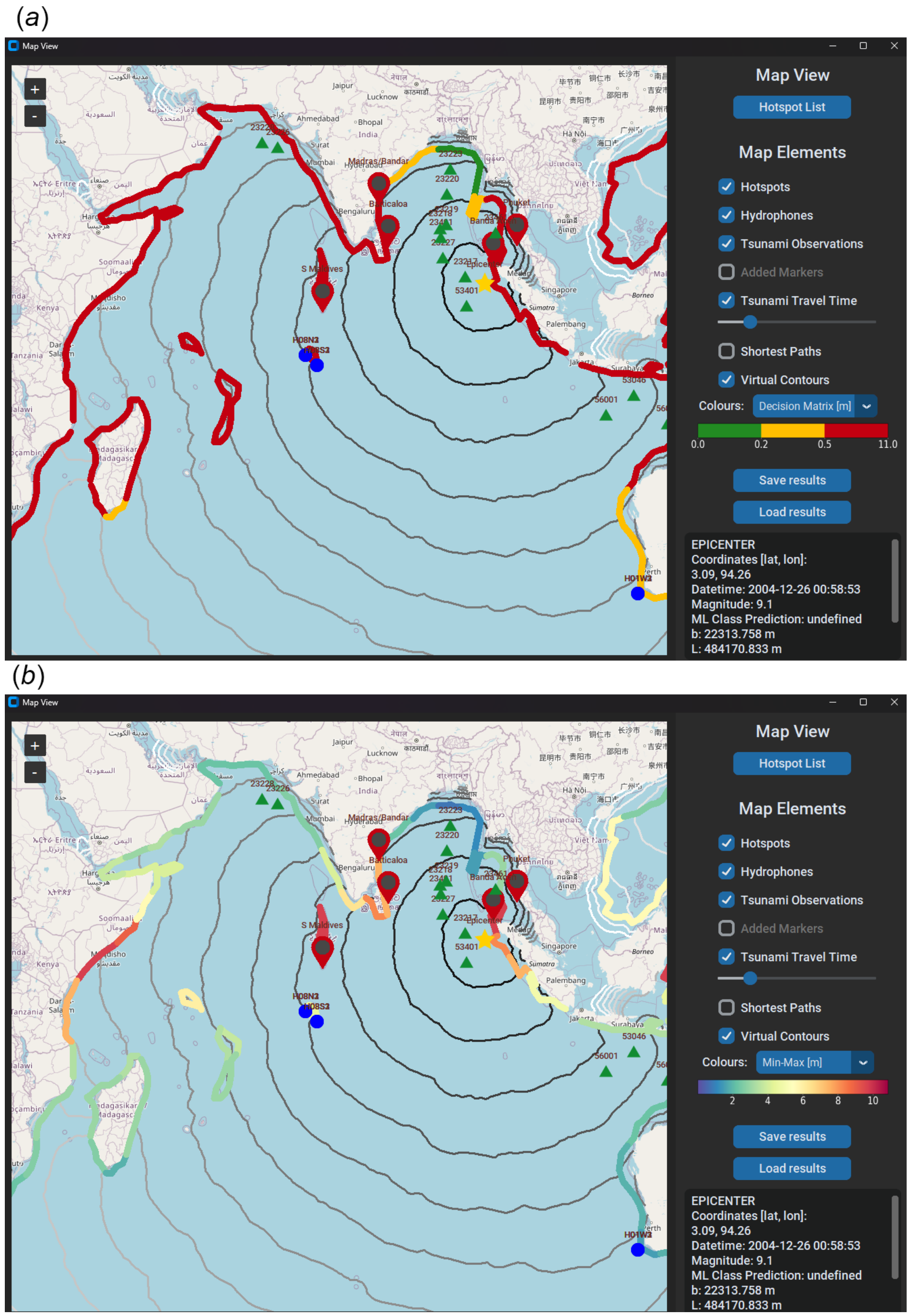

Fig. 4: Screenshots of the software (GREAT) for test case 2004 Mw 9.1 Sumatra earthquake and tsunami. Yellow star: earthquake epicentre. Green triangles: the location of current DART buoys. Blue circles: hydrophone stations H08S and H08N. Hotspots: user-defined points of interest (red for high risk, yellow for middle risk). (a) A snapshot from the software for showing tsunami arrival times (black contours), and size (coloured contours) at 50 m depth. (b) A snapshot from the software for showing tsunami evaluation contours at the coasts (red for high risk, yellow for middle risk/advisory, and green for no risk) at 50 m depth. Credit: reprinted from Figure 4 of Kadri et al. (2024).

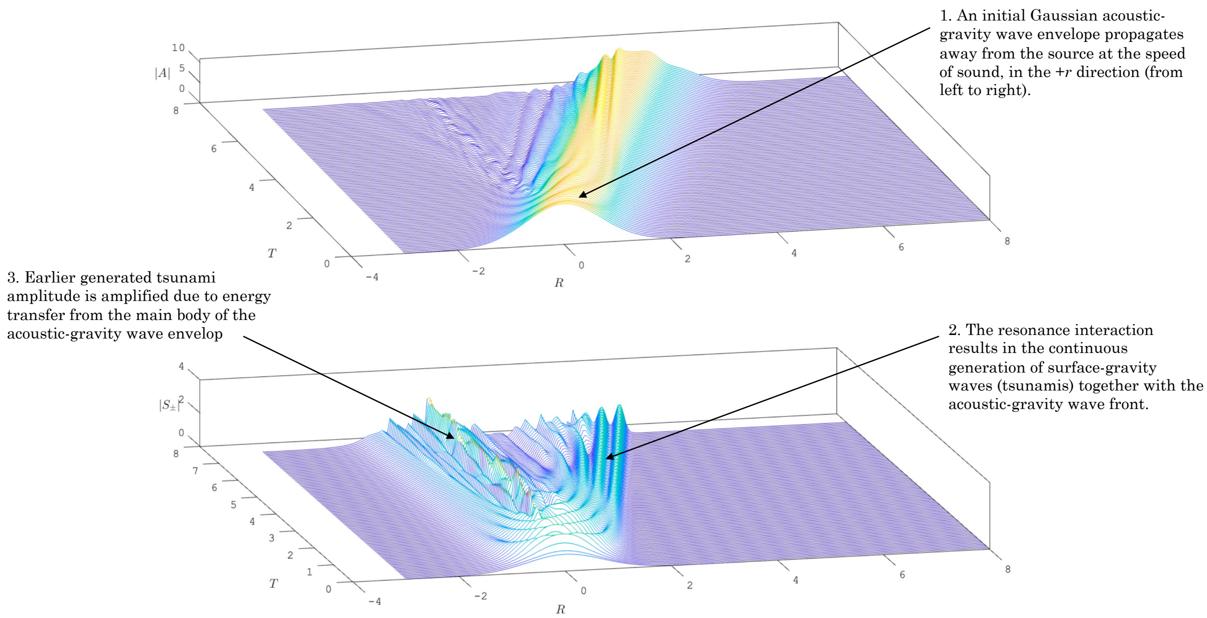

Fig. 5: Amplitude evolution of the acoustic (top) and main gravity (bottom) modes.

subsequent research by Kadri and Akylas (2016), Kadri and Stiassnie (2013) and others extending these concepts to include cubic nonlinearities.

4.2.1 Microseisms

These studies have implications for the generation of microseisms and reveal that in specific conditions, i.e., when two almost identical gravity waves interact with a single acoustic wave, significant energy transfer can occur between the wave modes. This interaction is primarily controlled by the spatial and temporal scales of the waves and the degree of nonlinearity involved. Kadri and Akylas (2016) demonstrated that resonant triad interactions could efficiently transfer nearly all energy from gravity waves to acoustic modes, especially in finely tuned scenarios (see Figure 2).

4.2.2 Meteotsunamis

The Tonga tsunami of January 2022, triggered by the Hunga Tonga–Hunga Ha'apai volcanic eruption, provided a unique case to study the nonlinear triad interactions of AGWs and their contribution to tsunami formation and propagation. The eruption generated powerful atmospheric AGWs that coupled with the ocean surface, leading to the formation and amplification of tsunami waves. Specifically, AGWs generated by the volcanic explosion interacted with the generated tsunami, enabling significant energy transfer from atmospheric waves to the tsunami. This interaction created a resonance effect that amplified tsunami wave heights. The study by Omira et al. argued that this coupling resulted in faster-than-expected tsunami travel times and higher wave amplitudes in distant coastal regions, unlike conventional point-sourced tsunamis triggered by earthquakes (Figure 5).

The nonlinear triad interactions involved in this process can be described as a resonance between two gravity waves and an acoustic wave. Such interactions are efficient in converting energy from one wave type to another, leading to amplified waves that travel with minimal energy loss. This phenomenon explains the unusual speed, global reach, and prolonged duration of the Tonga tsunami, making it a distinctive example of atmospheric-driven tsunami dynamics where nonlinear AGW interactions are fundamental.

4.2.3 Tsunami mitigation and energy harnessing

Tsunami mitigation using nonlinear triad resonance comprising a surface gravity wave and two acoustic waves, the evolution of the wave amplitudes is expressed in the form

$$\frac{\partial S}{\partial T} = -\beta A_1 A_2^* - \mathrm{i}\gamma S|S|^2 \tag{17}$$

for the tsunami, and

$$\frac{\partial A_1}{\partial T} - \mathrm{i}\alpha_1 \frac{\partial^2 A_1}{\partial X^2} = \mathrm{i}\beta_1 A_1 + \gamma_1 A_2^* S; \qquad \frac{\partial A_2}{\partial T} - \mathrm{i}\alpha_2 \frac{\partial^2 A_2}{\partial X^2} = \mathrm{i}\beta_2 A_2 + \gamma_2 A_1^* S \tag{18}$$

for the acoustic modes, where S is the amplitude of the tsunami, A_1 and A_2 are the amplitudes of the two acoustic waves, and α, β and γ are coupling coefficients.

Here, β_1 and β_2 are other coupling constants that depend on the interaction strength between the acoustic and gravity waves. These equations describe how the amplitudes of the tsunami and AGWs evolve over time through energy exchange. As a result, it could reduce the amplitude of the tsunami and redistribute its energy across a larger space to lower its impact on shorelines (Kadri 2017).

AGWs are capable of transporting significant amounts of energy over long distances: the resonant triad interaction offers the possibility of harnessing that energy. By carefully tuning the interaction parameters, some of the energy withdrawn from the tsunami (or surface-gravity waves in general) can be captured and used, possibly for generating power. However, this remains a challenging task, as generating AGWs at the required scales would need substantial energy inputs and a new technology.

5 Concluding remarks

In conclusion, acoustic-gravity wave (AGW) theory is an emerging field that provides critical understanding to a broad spectrum of geophysical phenomena, from microseisms and tsunamis to energy transfer mechanisms in oceanic and atmospheric environments. Through this brief review paper, we have highlighted some of the theoretical frameworks governing AGW propagation, including the effects of compressibility, elasticity, and nonlinear interactions. The advancement of linear AGW theory has provided practical solutions for real-time tsunami assessment, as well as for locating objects impacting the sea surface, such as meteorites and airplanes, or detecting underwater explosions and implosions, like the case of the Argentinian submarine ARA San Juan, which vanished off the coast of Argentina on November 15, 2017. The inclusion of elasticity in wave models is found to be essential in accurately describing the propagation of AGWs (and tsunami) over elastic seabeds, revealing the complexities in phase speed and energy transfer. Additionally, nonlinear mechanisms, such as the resonant triad interaction, play a key role in energy redistribution, offering potential for microseisms, meteotsunami amplification, tsunami mitigation and even energy harnessing. Despite these advancements, there remain challenges in translating theoretical concepts into practical technologies, particularly for energy capture from AGWs, which will require further technological innovation and research.

Future research should continue to focus on refining models of AGW propagation and interaction, with an emphasis on incorporating real-time data to enhance early warning systems and further explore the possibilities of AGW-based energy solutions. The parallels between classical AGW dynamics and quantum analogues, such as those seen in pilot-wave theory, remain largely unexplored but hold the potential to open exciting interdisciplinary opportunities. This suggests that AGWs could serve as a natural bridge between fluid mechanics, geophysics, and emerging quantum technologies. As our understanding deepens, the practical applications of AGWs will undoubtedly expand, addressing wave propagation challenges, advancing future technologies, and exploring fundamental scientific questions.

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